Week 4 — Forecasting with ARMA Models

Estimation, Model Selection, and Forecast Evaluation

Dr Juan Zurita

Economic & Business Forecasting
The University of Edinburgh · School of Economics

Estimation

Maximum Likelihood for ARMA

For ARMA(p, q): $y_t = c + \sum \phi_i y_{t-i} + \varepsilon_t + \sum \theta_j \varepsilon_{t-j}$

Under normality:

$$\ell(\theta) = -\frac{T}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\left[\log \sigma_t^2 + \frac{(y_t - \hat{y}_{t|t-1})^2}{\sigma_t^2}\right]$$

The Kalman filter computes $\hat{y}_{t|t-1}$ and σ_t^2 recursively.

Model Selection

Information Criteria in Practice

- Fit all ARMA(p, q) for $p = 0, \ldots, P$ and $q = 0, \ldots, Q$
- Compute AIC and BIC for each
- Select the model with the lowest criterion value

Rule of thumb

AIC tends to select larger models (better forecasts).
BIC tends to select the true model (if it's in the set).

Forecasting

h-Step-Ahead Forecasts

The minimum MSE forecast is the conditional mean:

$$\hat{y}_{T+h|T} = E[y_{T+h} \mid y_1, \ldots, y_T]$$

AR(1) example:

$$\hat{y}_{T+1|T} = c + \phi y_T \tag{1}$$

$$\hat{y}_{T+2|T} = c + \phi \hat{y}_{T+1|T} = c(1 + \phi) + \phi^2 y_T \tag{2}$$

$$\hat{y}_{T+h|T} \to \frac{c}{1 - \phi} = \mu \quad \text{as } h \to \infty \tag{3}$$

Forecasts revert to the mean at rate ϕ.

Forecast Uncertainty

The forecast error variance grows with horizon:

$$\text{Var}(y_{T+h} - \hat{y}_{T+h|T}) = \sigma^2 \sum_{j=0}^{h-1} \psi_j^2$$

where ψ_j are MA(∞) coefficients.

Confidence intervals:

$$\hat{y}_{T+h|T} \pm z_{\alpha/2} \sqrt{\text{Var}(e_{T+h|T})}$$

Diagnostics

Residual Checking

If the model is correct, residuals $\hat{\varepsilon}_t$ should be white noise:

1. **Plot** residuals over time: no patterns?
2. **ACF of residuals**: all within confidence bands?
3. **Ljung-Box test**: $Q(m) = T(T+2)\sum_{k=1}^{m} \frac{\hat{\rho}_k^2}{T-k} \sim \chi^2(m-p-q)$
4. **Normality**: Q-Q plot, Jarque-Bera test

Forecast Evaluation

Metric	Formula
RMSE	$\sqrt{\frac{1}{H}\sum(y_{T+h}-\hat{y}_{T+h})^2}$
MAE	$\frac{1}{H}\sum\lvert y_{T+h}-\hat{y}_{T+h}\rvert$
MAPE	$\frac{100}{H}\sum\left\lvert\frac{y_{T+h}-\hat{y}_{T+h}}{y_{T+h}}\right\rvert$

Always evaluate on out-of-sample data. Rolling-window or expanding-window evaluation is best practice.

Summary

- Estimate ARMA by **MLE**; select with **AIC/BIC**
- Forecasts **revert to the mean**; uncertainty grows with horizon
- Check residuals for white noise
- Evaluate on out-of-sample data
- Next week: exponential smoothing