Week 7 — State Space Models & the Kalman Filter

Unobserved Components and Optimal Filtering

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State Space Representation

The General Model

State equation:

$$\boldsymbol{\alpha}_{t+1} = \mathbf{T}\boldsymbol{\alpha}_t + \mathbf{R}\boldsymbol{\eta}_t, \quad \boldsymbol{\eta}_t \sim N(\mathbf{0}, \mathbf{Q})$$

Observation equation:

$$\mathbf{y}_t = \mathbf{Z}\boldsymbol{\alpha}_t + \boldsymbol{\varepsilon}_t, \quad \boldsymbol{\varepsilon}_t \sim N(\mathbf{0}, \mathbf{H})$$

- $\boldsymbol{\alpha}_t$: unobserved state vector
- $\mathbf{y}_t$: observed data
- Many time-series models are special cases

Why State Space?

Unifying framework:

Model	**State Space Form**
ARMA	state = lagged values/errors
Exponential smoothing	state = level, trend, seasonal
Structural time series	state = trend + season + cycle
Dynamic factor model	state = latent factors

The Kalman Filter

Prediction and Update

Prediction step:

$$\hat{\alpha}_{t|t-1} = \mathbf{T}\hat{\alpha}_{t-1|t-1} \tag{1}$$

$$\mathbf{P}_{t|t-1} = \mathbf{T}\mathbf{P}_{t-1|t-1}\mathbf{T}' + \mathbf{R}\mathbf{Q}\mathbf{R}' \tag{2}$$

Update step:

$$\mathbf{K}_t = \mathbf{P}_{t|t-1}\mathbf{Z}'(\mathbf{Z}\mathbf{P}_{t|t-1}\mathbf{Z}' + \mathbf{H})^{-1} \tag{3}$$

$$\hat{\alpha}_{t|t} = \hat{\alpha}_{t|t-1} + \mathbf{K}_t(\mathbf{y}_t - \mathbf{Z}\hat{\alpha}_{t|t-1}) \tag{4}$$

$\mathbf{K}_t$ is the Kalman gain — optimal weighting of new information.

Structural Time Series

Local Level Model

$$y_t = \mu_t + \varepsilon_t \qquad \varepsilon_t \sim N(0, \sigma_\varepsilon^2) \tag{5}$$

$$\mu_{t+1} = \mu_t + \eta_t \qquad \eta_t \sim N(0, \sigma_\eta^2) \tag{6}$$

- Signal-to-noise ratio: $q = \sigma_\eta^2 / \sigma_\varepsilon^2$
- $q \to 0$: constant level (all variation is noise)
- $q \to \infty$: random walk (all variation is signal)

Local Linear Trend

Adds a stochastic slope:

$$\mu_{t+1} = \mu_t + \nu_t + \eta_t \tag{7}$$

$$\nu_{t+1} = \nu_t + \zeta_t \tag{8}$$

Can add seasonal, cycle, and regression components.

Next week: dynamic factor models for large datasets.