Week 2 — Unconstrained Optimisation I

Gradients, Hessians & Convexity

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First-Order Conditions

For $f : \mathbb{R}^n \to \mathbb{R}$, the **gradient** is $\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)^T$.

First-Order Necessary Condition

If $\mathbf{x}^*$ is a local extremum of f and f is differentiable, then $\nabla f(\mathbf{x}^*) = \mathbf{0}$.

The Hessian Matrix

$$H_f(\mathbf{x}) = \begin{pmatrix} f_{x_1 x_1} & f_{x_1 x_2} \\ f_{x_2 x_1} & f_{x_2 x_2} \end{pmatrix}$$

Second-Order Conditions at $\nabla f = 0$

- H pos. def. $\Rightarrow$ local min
- H neg. def. $\Rightarrow$ local max
- H indefinite $\Rightarrow$ saddle

Convexity

f is **convex** if the line segment between any two points lies above the graph:
$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$

Key Result

If f is convex, every local minimum is a global minimum.

Characterisation: f convex $\iff$ $H_f \succeq 0$ everywhere.

Economic Application

Profit maximisation: $\pi(q) = pq - C(q)$

FOC: $p = C'(q)$ (price = marginal cost)

SOC: $C''(q) > 0$ (increasing marginal cost)

If cost is convex, the profit function is concave in q — the FOC is sufficient.

Summary

- Gradient = direction of steepest ascent; zero at optima
- Hessian classifies critical points via definiteness
- Convexity: local = global — the gold standard
- **Next week:** algorithms that find the optimum