Week 4 — Constrained Optimisation I

Lagrange Multipliers & Economic Applications

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The Constrained Problem

Maximise $f(\mathbf{x})$ **subject to** $g(\mathbf{x}) = 0$

Lagrangian: $\mathcal{L}(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda\, g(\mathbf{x})$

FOC: $\nabla f = \lambda \nabla g$ and $g(\mathbf{x}) = 0$

Geometrically: at the optimum, the level curve of f is tangent to the constraint.

Utility Maximisation

$\max \; u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$ s.t. $p_1 x_1 + p_2 x_2 = m$

Marshallian demands: $x_1^* = \frac{\alpha m}{p_1}$, $x_2^* = \frac{(1-\alpha) m}{p_2}$

The multiplier $\lambda^* =$ marginal utility of income.

The Envelope Theorem

Envelope Theorem

$\frac{\partial V}{\partial m} = \lambda^*$

The Lagrange multiplier measures the shadow value of the constraint.

An extra $1 of income raises utility by exactly λ^*.

Bordered Hessian

To verify a constrained maximum, check the **bordered Hessian**:

$$\bar{H} = \begin{pmatrix} 0 & g_{x_1} & g_{x_2} \\ g_{x_1} & \mathcal{L}_{x_1x_1} & \mathcal{L}_{x_1x_2} \\ g_{x_2} & \mathcal{L}_{x_2x_1} & \mathcal{L}_{x_2x_2} \end{pmatrix}$$

For a maximum with one constraint: $|\bar{H}| > 0$.

Summary

- Lagrange multipliers handle equality constraints
- $\lambda^* =$ shadow price of the constraint
- The envelope theorem connects multipliers to comparative statics
- **Next week:** inequality constraints and KKT conditions