Week 5 — Constrained Optimisation II

KKT Conditions & Inequality Constraints

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Inequality Constraints

Most economic constraints are **inequalities**: budgets, non-negativity, capacity.

$\min f(\mathbf{x})$ s.t. $g_i(\mathbf{x}) \leq 0,\ i = 1, \ldots, m$

KKT Conditions

1. **Stationarity:** $\nabla f + \sum \mu_i \nabla g_i = 0$
2. **Primal feasibility:** $g_i(\mathbf{x}) \leq 0$
3. **Dual feasibility:** $\mu_i \geq 0$
4. **Complementary slackness:** $\mu_i \, g_i(\mathbf{x}) = 0$

Under convexity, KKT conditions are sufficient.

Complementary Slackness

$\mu_i \, g_i = 0$ means:

- Constraint **binding** ($g_i = 0$): multiplier can be positive
- Constraint **slack** ($g_i < 0$): multiplier must be zero

Economics: a resource has a positive shadow price only if it is fully used.

Portfolio Choice

$\min \ \mathbf{w}^T \Sigma \mathbf{w}$ s.t. $\mathbf{w}^T \boldsymbol{\mu} \geq r_{\min}$, $\sum w_i = 1$, $w_i \geq 0$

No short-selling constraints are **inequalities**.

KKT tells us: assets with zero weight have insufficient expected return to justify inclusion.

Summary

- KKT $=$ Lagrange multipliers extended to inequalities
- Complementary slackness: binding vs. slack constraints
- Under convexity, KKT is necessary *and* sufficient
- **Next week:** convex optimisation theory and CVXPY