Week 8 — Dynamic Optimisation I

Optimal Control & the Maximum Principle

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Why Dynamic Optimisation?

Economic decisions are **intertemporal**: save, invest, consume over time.

We need to optimise *paths* $\{x(t)\}$, not just points.

Two frameworks:

1. Calculus of variations / optimal control (continuous time)
2. Dynamic programming (discrete time — next week)

The Euler–Lagrange Equation

max $\int_{t_0}^{t_1} F(t, y, \dot{y})\, dt$

Necessary condition:

$$\frac{\partial F}{\partial y} - \frac{d}{dt}\frac{\partial F}{\partial \dot{y}} = 0$$

Pontryagin's Maximum Principle

Hamiltonian: $H = f(x, u, t) + \lambda g(x, u, t)$

Necessary conditions:

1. $\partial H / \partial u = 0$ (optimality)
2. $\dot{\lambda} = -\partial H / \partial x$ (costate)
3. $\dot{x} = \partial H / \partial \lambda$ (state)
4. Transversality condition

The Ramsey Model

max $\int_0^\infty e^{-\rho t} u(c)\, dt$ s.t. $\dot{k} = f(k) - \delta k - c$

Euler equation (Keynes–Ramsey rule):

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}[f'(k) - \delta - \rho]$$

Consumption grows when the marginal product of capital exceeds the discount rate plus depreciation.

Summary

- Dynamic optimisation: choosing optimal paths over time
- Euler–Lagrange equation: necessary condition for variational problems
- Maximum Principle: powerful extension to control problems
- The Ramsey model: foundation of modern growth theory
- **Next week:** dynamic programming and the Bellman equation