Lecture 7: Function Approximation

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Where we're going

The Neoclassical Growth Model

- Suppose we want to solve the problem:

$$
\begin{aligned}
V(k,z) = \max_{c,k',n} \quad & u(c,n) + \beta \mathbb{E}\left[V(k',z') \mid z\right] & \\
\text{s.t.} \quad & c + k' = zF(k,n) & \text{Resource Constraint} \\
& \log(z') = \rho \log(z) + \epsilon & \log(z) \text{ is an } AR(1) \\
& \epsilon \sim N(0,\sigma) & \text{Shocks to } z \text{ are log-normal}
\end{aligned}
\tag{1}
$$

  where

  - c is consumption
  - k is capital, and r is the rental price of capital
  - n is labor supply, and w is the wage
  - F is a constant returns to scale production function
  - β, ρ and σ are paramters

- If we can't get a solution by hand, then what does "solve this problem" even mean?

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What is a "solution" in quantitative economics?

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- We will see next week that there is a unique function $V : \mathbb{R}^2 \to \mathbb{R}$ that satisfies eq. (1)
- In general, however, we cannot get an exact formula for V (a "closed-form solution")
- We have to settle for finding an approximation of V: call it $\widehat{V}$
  - As long as the solution to eq. (1) is unique, if we find an approximation $\widehat{V}$ that also satisfies it, then we can call it a day
  - It turns out that if we repeatedly solve the maximization problem above, starting from an initial guess and updating $\widehat{V}$ each iteration, we can be sure that we will converge to the true solution
  - This process, called **value function iteration** is what we will be learning about next week

This week: Function Approximation

- This week, we will be focusing on different methods to approximate V with some other function $\widehat{V}$
- The key questions we'll be answering:
  1. What does it mean to say that $\widehat{V}$ is "close" to V (i.e, that it approximates it well)
  2. What kinds of approximations work well in practice?
  3. How do we represent these approximations on a computer?
  4. How can we calculate them efficiently?

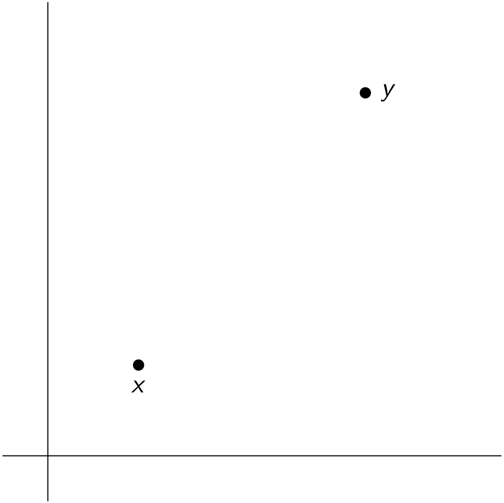
Section 1

Distance, Functions, and the Generalized Dot Product

How do we measure distance in $\mathbb{R}^n$?

- Suppose I have two points in $\mathbb{R}^n$: x and y
- How do I measure how far apart they are?
- Pythagorean theorem says: draw the corresponding right triangle, and use
$$a^2 + b^2 = c^2$$
- In this case
$$c^2 = (y_1 - x_1)^2 + (y_2 - x_2)^2$$
- This happens to correspond nicely with the norm of the difference between these two points:
$$c^2 = ||y - x||^2 = (y - x) \cdot (y - x)$$

Recall that $||x||^2 := x \cdot x = \sum_{i=1}^n x_i^2$



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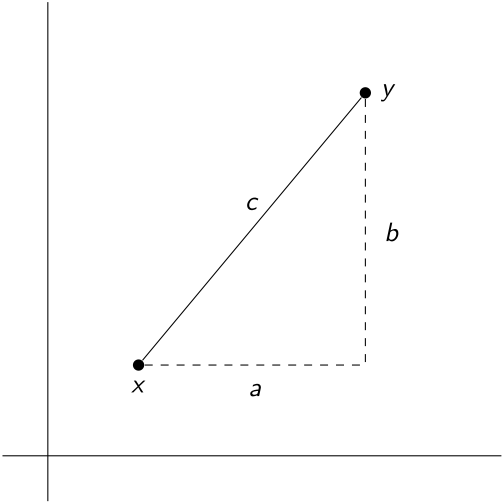

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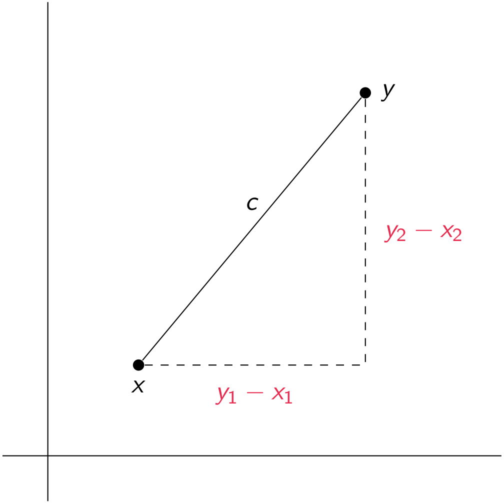

The dot product

- We've already seen the dot product show up:

$$x \cdot y := \sum_{i=1}^{n} x_i y_i$$

- It has this natural connection to our notion of distance:

$$||y - x||^2 = \sum_{i=1}^{n} (y_i - x_i)^2 = (y - x) \cdot (y - x)$$

- It is integral to what matrix multiplication looks like:

$$\begin{bmatrix} — & a_1 & — \\ — & a_2 & — \\ & \vdots & \\ — & a_n & — \end{bmatrix} \begin{bmatrix} | & | & & | \\ b_1 & b_2 & \dots & b_k \\ | & | & & | \end{bmatrix} = \begin{bmatrix} a_1 \cdot b_1 & a_1 \cdot b_2 & \dots & a_1 \cdot b_k \\ a_2 \cdot b_1 & a_2 \cdot b_2 & \dots & a_2 \cdot b_k \\ \vdots & \vdots & \ddots & \vdots \\ a_n \cdot b_1 & a_n \cdot b_2 & \dots & a_n \cdot b_k \end{bmatrix}$$

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The dot product encodes the angle between two vectors

- Suppose we have two vectors x and y that lie on the unit circle ($||x|| = ||y|| = 1$)
- We know that both vectors are defined (in polar coordinates) by their angles:

$$x = (\cos\theta_x, \sin\theta_x) \qquad y = (\cos\theta_y, \sin\theta_y)$$

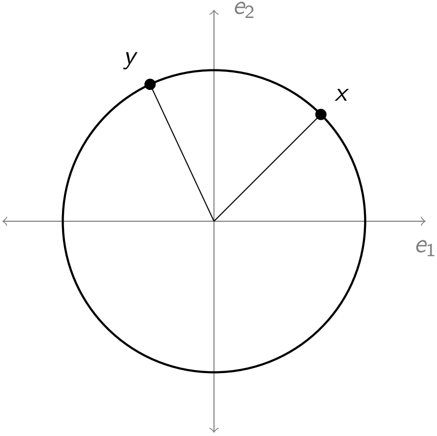
- Recall the cosine subtraction formula:

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \qquad (2)$$

- That means that the dot product is just:

$$\begin{aligned} x \cdot y &= \cos\theta_x \cos\theta_y + \sin\theta_x \sin\theta_y && \text{Def of dot product} \\ &= \cos(\theta_y - \theta_x) && \text{By eq. (2)} \end{aligned}$$

- So we know that $x \cdot y = 0$ if and only if the **cosine of the angle between them is zero**. (I.e, the vectors are orthogonal)



This generalizes to when x and y are not on the unit circle, as well as to $\mathbb{R}^n$. In the general case:

$$x \cdot y = ||x|| \times ||y|| \times \cos\theta$$

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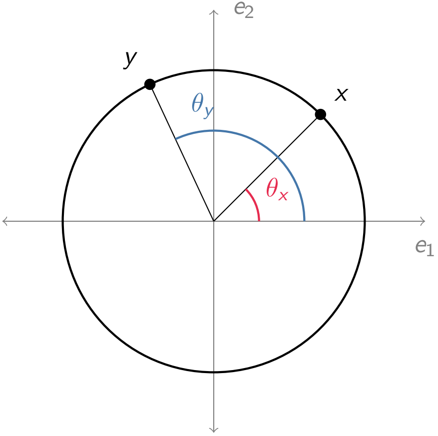

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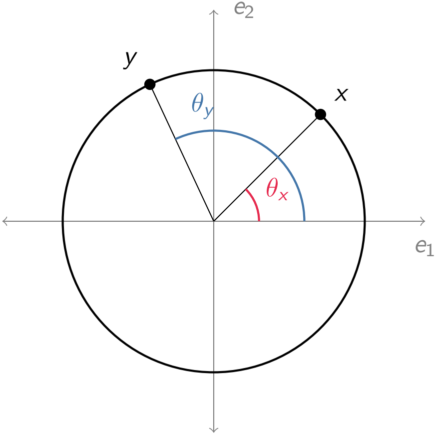

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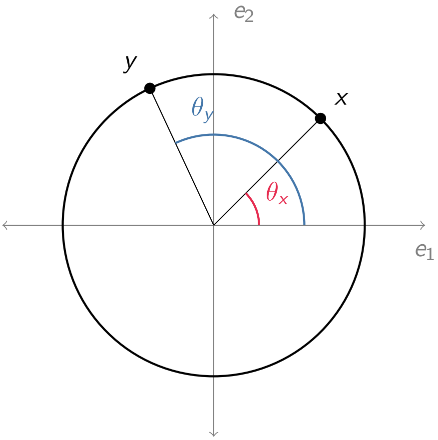

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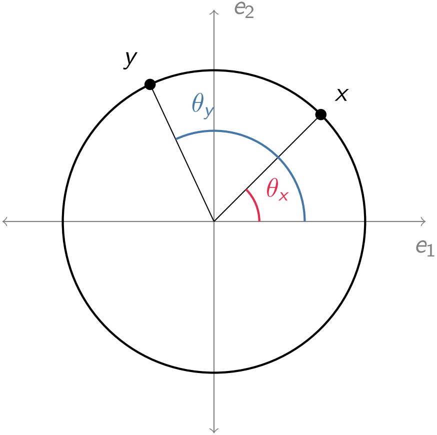

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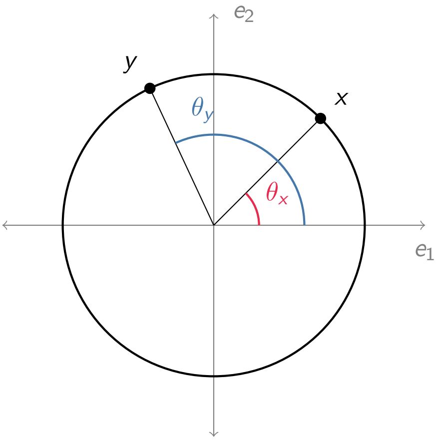

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How can we represent functions on a computer?

- If we don't have an explicit formula, we can try encoding the function values on a grid of points
- Let $f(x) = \sin(x)$, and consider a grid with $n + 1$ points:

$$X_n = \left\{ \frac{2\pi i}{n} \,\middle|\, i = 0, 1, \ldots, n \right\}$$

- For every point x_i, we save a corresponding

$$\widehat{f}_i^n = \sin\left(\frac{2\pi i}{n}\right)$$

- Maybe we imagine that if we're off grid, we will interpolate linearly (we'll talk about this later)
- We hope that if n gets large, this is going to be good enough...

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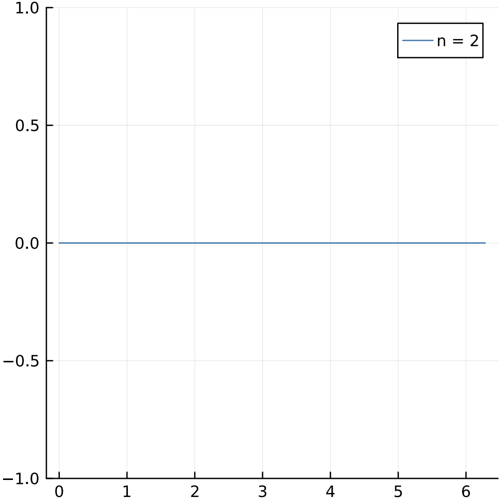

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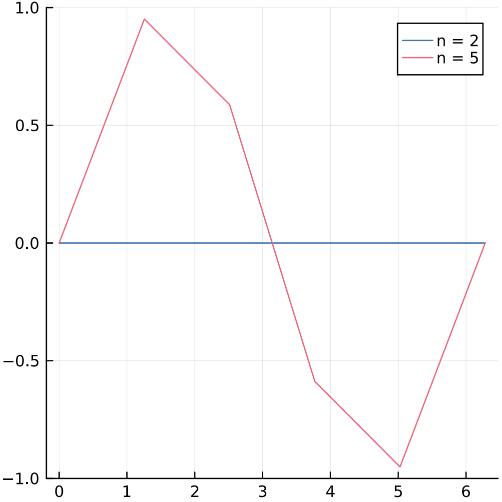

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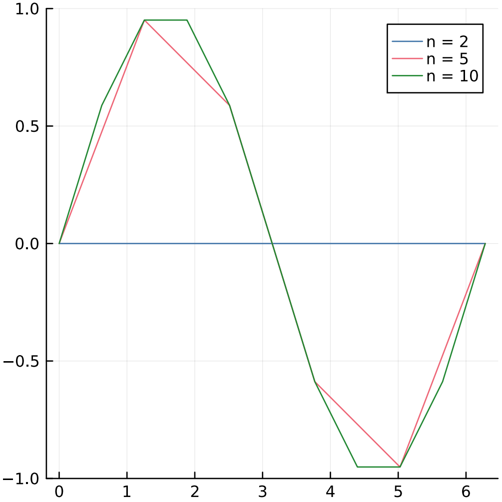

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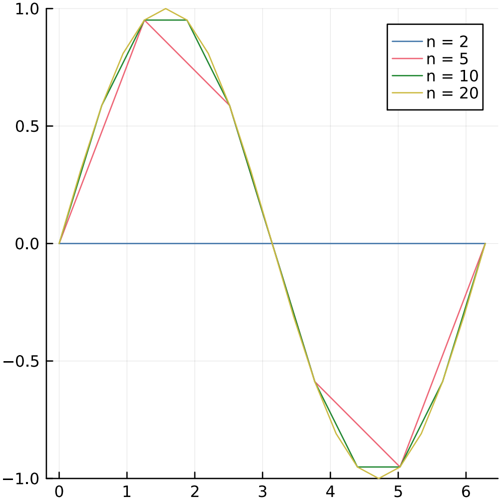

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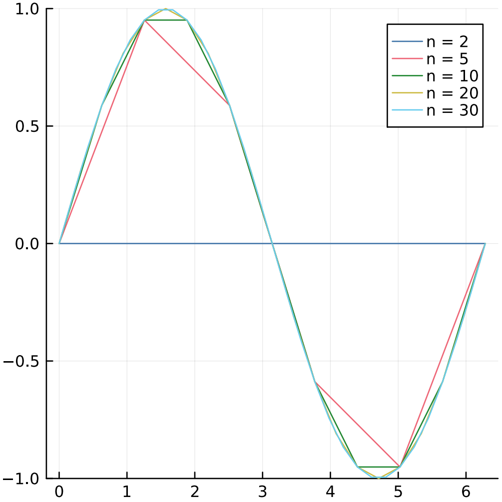

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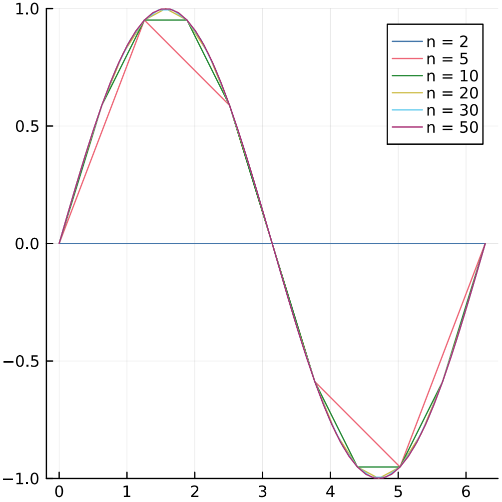

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What is the “length” of a function?

Consider a function f on [0,1], and think about f as the vector $\widehat{f}^n \in \mathbb{R}^{n+1}$ with

$$X_n = \left\{ \frac{i}{n} \;\middle|\; i = 0, \ldots, n \right\}$$

- Let’s try a definition of $\|f\|$:

$$\|f\|^2 := \lim_{n\to\infty} \frac{1}{n} \|\widehat{f}^n\| \tag{3}$$

We have to divide by n because we’re increasing the number of dimensions we’re summing over.

- Define $\Delta x_n = \frac{1}{n}$. We can write this as:

$$\|f\|^2 = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \left(\widehat{f}_i^n\right)^2 = \underbrace{\lim_{n\to\infty} \sum_{i=0}^{n} f(x_i)^2 \Delta x_n}_{\text{A Riemann Sum}} = \int_0^1 f(x)^2 dx$$

This works on more general domains as well (not just [0,1]) as well as for nonuniform grids

What is the "length" of a function?

Consider a function f on [0,1], and think about f as the vector $\widehat{f}^n \in \mathbb{R}^{n+1}$ with

$$X_n = \left\{ \frac{i}{n} \;\middle|\; i = 0, \ldots, n \right\}$$

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- We can now define the distance between functions:

$$||f - g||^2 = \int_0^1 \Big(f(x) - g(x)\Big)^2 dx$$

- Suppose we have a function $f : [0,1] \to \mathbb{R}$, and a proposed approximation $\widehat{f} : [0,1] \to \mathbb{R}$.

- **Question:** How should we judge how "good" and approximation $\widehat{f}$ is?

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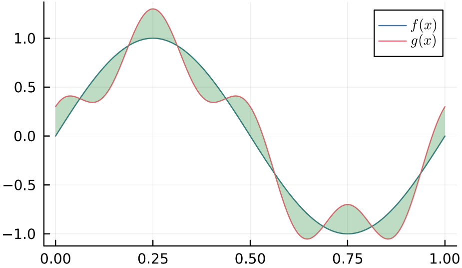

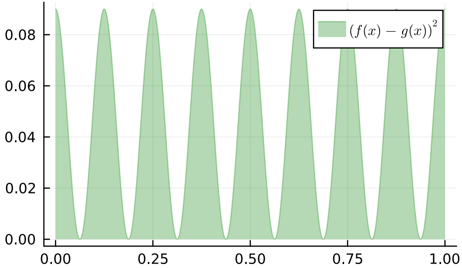

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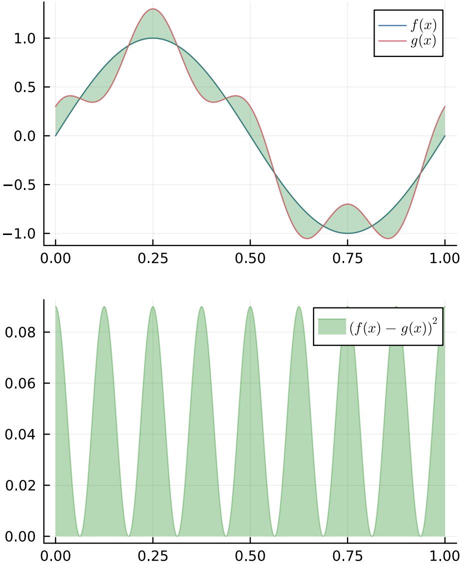

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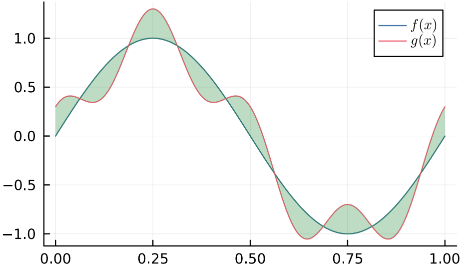
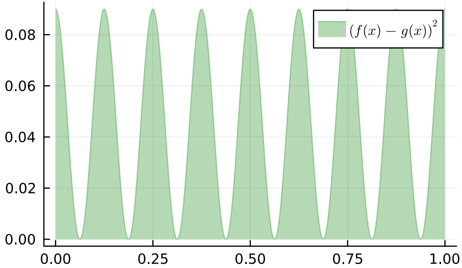




Can functions be orthogonal?

Not examinable

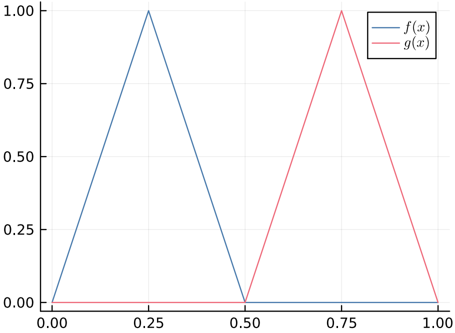
- We can define something like a "dot product" for functions:

$$\langle f, g \rangle := \int_0^1 f(x)g(x)dx$$

- This is called an **inner product**
- Just like with the dot product

$$\langle f, f \rangle = ||f||^2 = \int_0^1 f(x)^2 dx$$

- Even more importantly, if $\langle f, g \rangle = 0$, that means we can meaningfully say that these function are **orthogonal**



Notice that in this case, $f(x)g(x) = 0$ since one of the two functions is always zero. That means

$$\langle f, g \rangle = 0$$

and so f and g are orthogonal

Section 2

Interpolation with Global Polynomials

Interpolating a function

- Let $f : [0,1] \to \mathbb{R}$ be a continuous function
  - Suppose you've already been given a grid $X = \{x_i\}_{i=1}^n$ and the evaluated $y = \{y_i\}_{i=1}^n$ where $y_i = f(x_i)$
  - It's easy to approximate f on grid – we already calculated its values – but we want to be able to approximate f off of the grid without evaluating f any more times
- Let's look for a polynomial $p(x) = \sum_{s=0}^{n-1} \alpha_i x^s$ that approximates the function well.

  Notice that I've chosen a polynomial with as many coefficients as we have data points. If we want to fit our data exactly, we will need as many degrees of freedom as we have observations.
- It should:
  1. Fit our function exactly on the grid of x_i
  2. Approximate f well off-grid

     i.e, $||f - p||$ should be small, and ideally should approach zero as n increases
- This is called an **interpolation problem**

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The Vandermonde Matrix

- If we want $p(x_i) = y_i$ for all i, then that implies:

$$a_0 + a_1 x_i + a_2 x_i^2 + \cdots + a_{n-1} x^{n-1} = y_i \quad \text{for } i = 1, \ldots, n \tag{4}$$

- Notice that this is a linear system of equations in the coefficients a:

$$\underbrace{\begin{bmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \ldots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \ldots & x_n^{n-1} \end{bmatrix}}_{V} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \tag{5}$$

- V is called the **Vandermonde matrix**
- It turns out that the solution to this system is unique, so long as the x_i are distinct Proof
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Lagrange Interpolation in Practice

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function vandermonde(X)
    n = length(X)
    V = [xi^s for xi in X, s in 0:n-1]
    return V
end

lagrange(X,y)= vandermonde(X)\y

function evaluate(a, x)
    sum(a[s] * x^(s-1) for s in eachindex(a))
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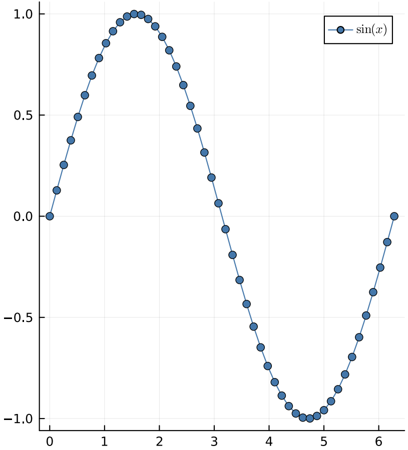

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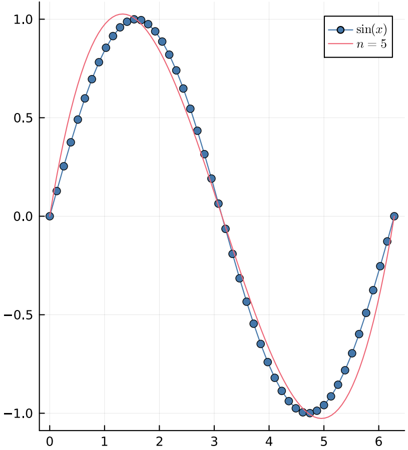

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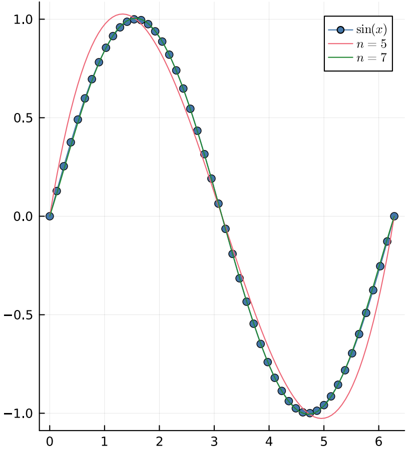

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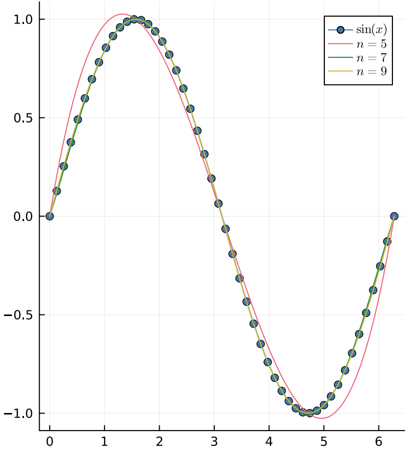
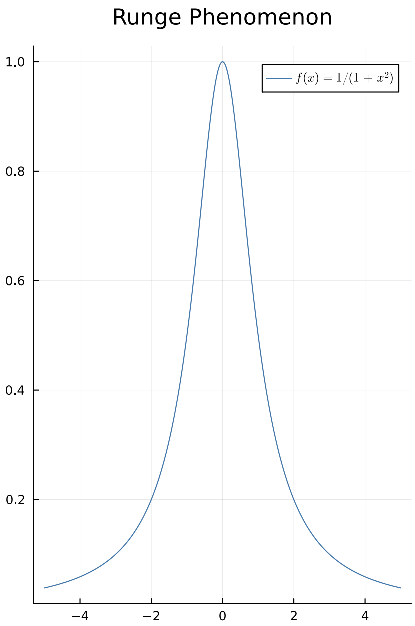


Lagrange Interpolation can fail badly

- So far, it seems Lagrange interpolation works well
- Unfortunately, there are a number of well known cases where it fails **catastrophically**
- Consider
$$f(x) = \frac{1}{1+x^2}$$
- When $n = 4$ the interpolant isn't great, but 4 points isn't that many
- By the time we're up to $n = 11$, it doesn't look like things are getting better
- In fact, you can show that this is a case where Lagrange interpolation **will never converge**
- Adding more data does not fix the problem. This is called the **Runge phenomenon**



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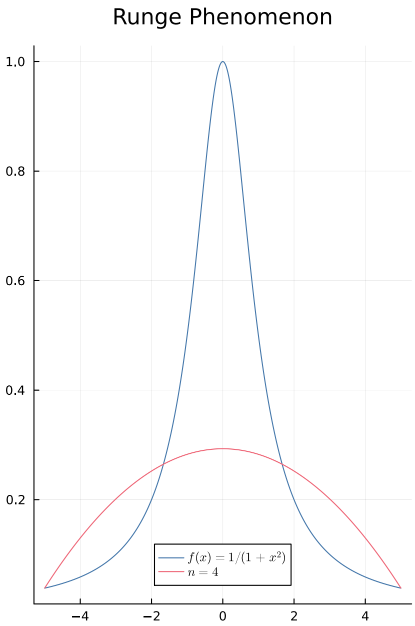

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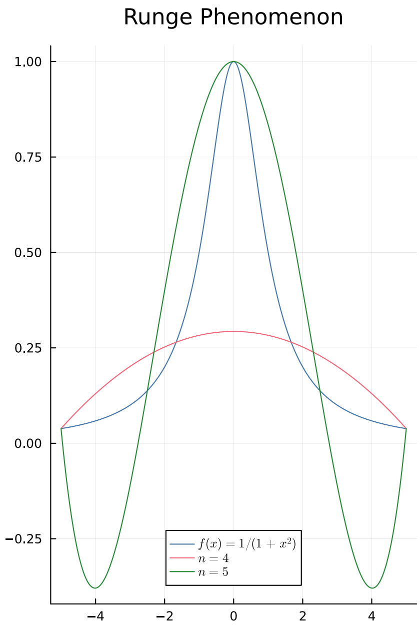

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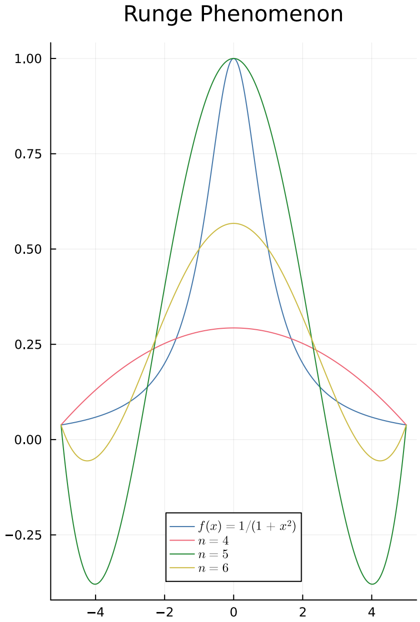

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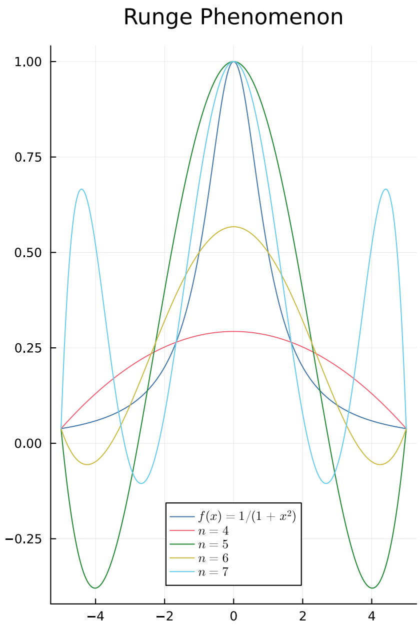

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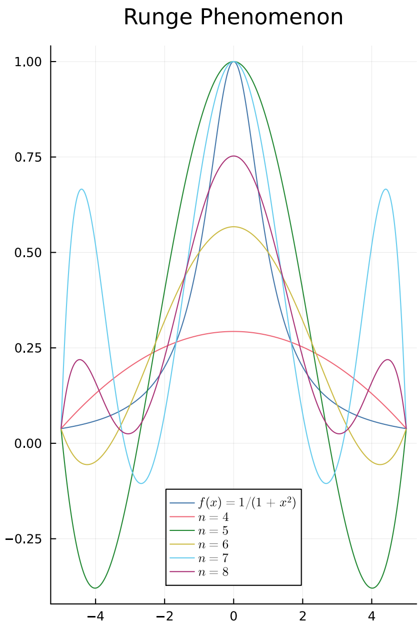

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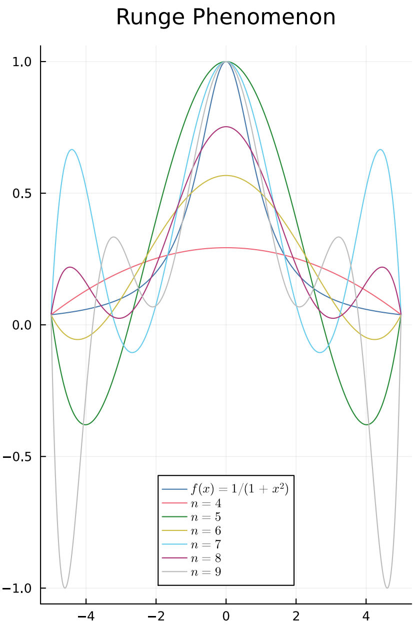

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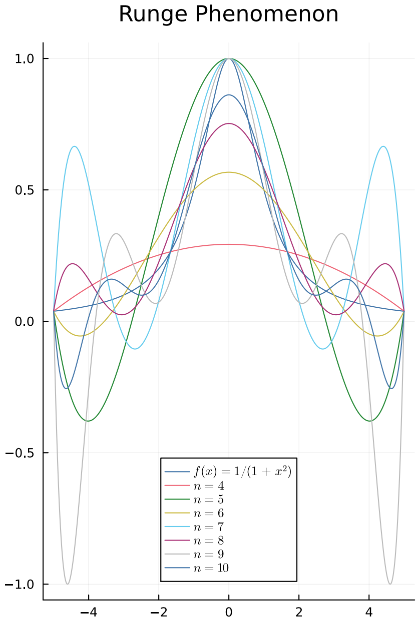

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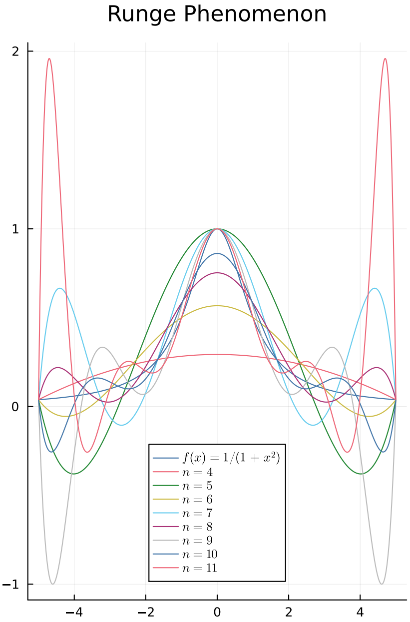

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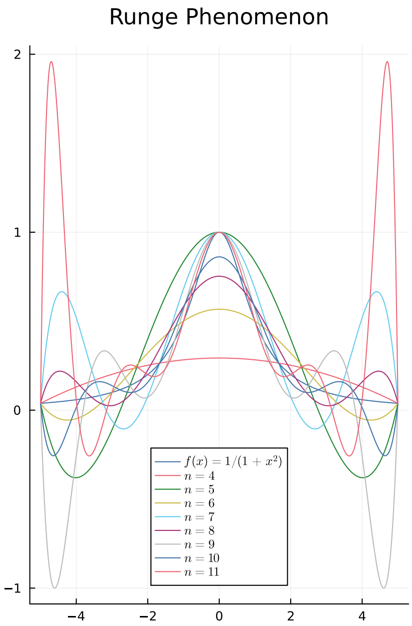

How to avoid the Runge phenomenon

- The Runge phenomenon (explosive oscillation at the edges) tends to occur in most polynomial interpolation schemes with *evenly spaced grids*
  - High order polynomial terms tend to grow explosively as x gets larger
  - When you try to hit the extra data points on the edge of the domain by adding a high order polynomial term like x^{11}, that induces even more oscillations elsewhere in the domain
- To avoid this, you can:
  1. use another family of smooth polynomials called **Chebyshev polynomials**
  2. use piecewise polynomials (Linear Interpolation, Splines, etc...)

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Chebyshev Polynomials

- Define $T_n(x) = \cos(n \cos^{-1} x)$ for $x \in [-1, 1]$
- The family of polynomials $\{T_n\}_{n=0}^{\infty}$ are called the **Chebyshev polynomials**
- Why are these actually polynomials?
  - You can show that these functions satisfy the formula (recurrence relationship):

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- Bounded between [-1, 1] so long as $x \in [-1, 1]$
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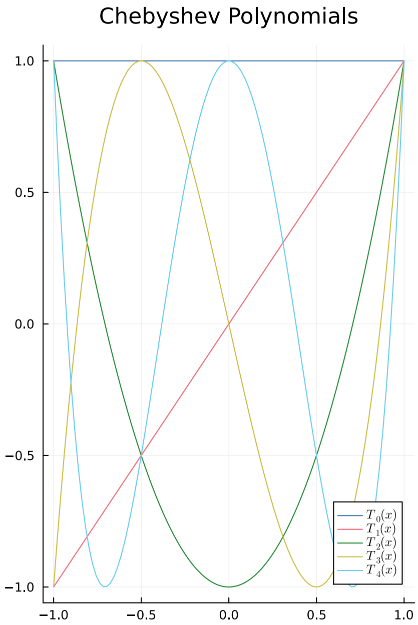
  Specifically (and not examinable), they are orthogonal with respect to an appropriate weighting function. I.e,

$$\int_{-1}^{1} T_n(x) T_k(x) w(x) dx = 0$$

  for $n \neq k$ and $w(x) = \frac{1}{\sqrt{1-x^2}}$
- You want nodes $\{x_k\}$ that are *unevenly spaced*.
- There are a known set of interpolation points that minimize the approximation error:

$$x_k = -\cos\left(\frac{2k-1}{2n}\pi\right) \quad \text{for } k = 1, \ldots, n$$

- **Chebyshev Approximation Theorem**: As long as our function f is smooth (has continuous kth derivatives for some $k \geq 1$) Chebyshev approximation converges "nicely" to f Theorem



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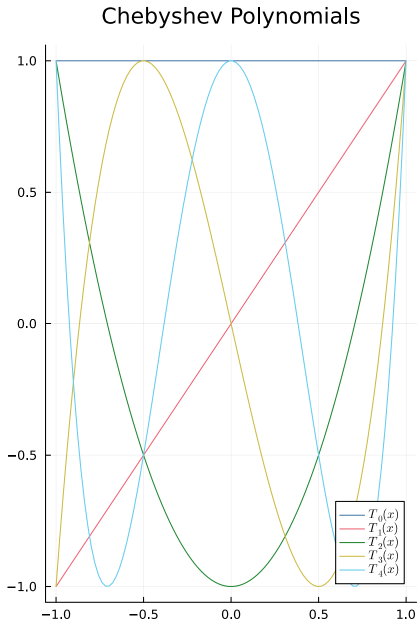

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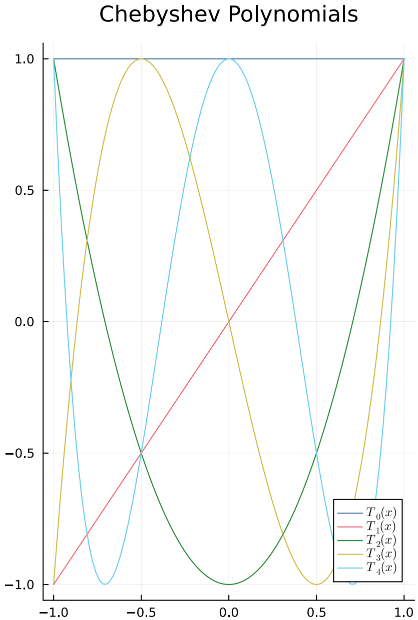

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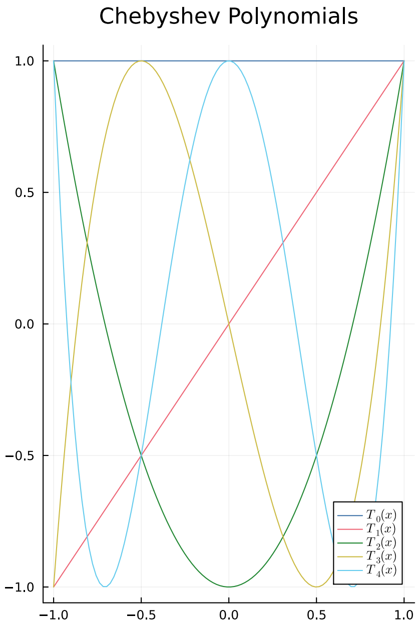

Chebyshev Regression (Approximation) Algorithm

We want an nth degree Chebyshev approximation:

1. Compute the $m \geq n+1$ Chebyshev interpolation nodes on $[-1, 1]$:
$$z_k = -\cos\left(\frac{2k-1}{2m}\pi\right) \quad k = 1, \ldots, m$$
2. For interpolation on $[a, b]$ instead of $[-1, 1]$, adjust the nodes to the appropriate interval:
$$x_k = (z_k + 1)\left(\frac{b-a}{2}\right) + a \quad k = 1, \ldots, m$$
3. Evaluate f at the appropriate points: $y_k = f(x_k)$ for $k = 1, \ldots, m$
4. Compute the Chebyshev coefficients:
$$c_i = \left(\frac{\sum_{k=1}^{m} y_k T_i(z_k)}{\sum_{k=1}^{m} T_i(z_k)^2}\right)$$
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$$\hat{f}(x) = \sum_{i=0}^{n} c_i T_i\left(2\frac{x-a}{b-a} - 1\right) \tag{7}$$

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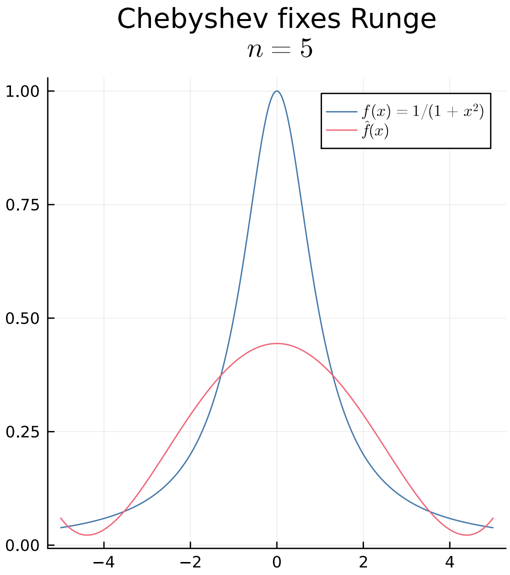

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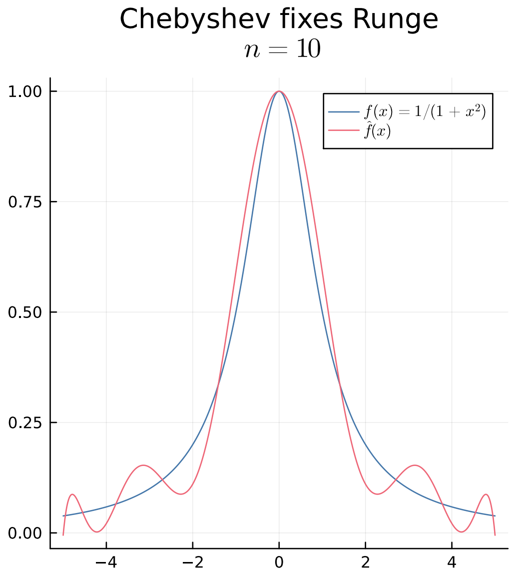

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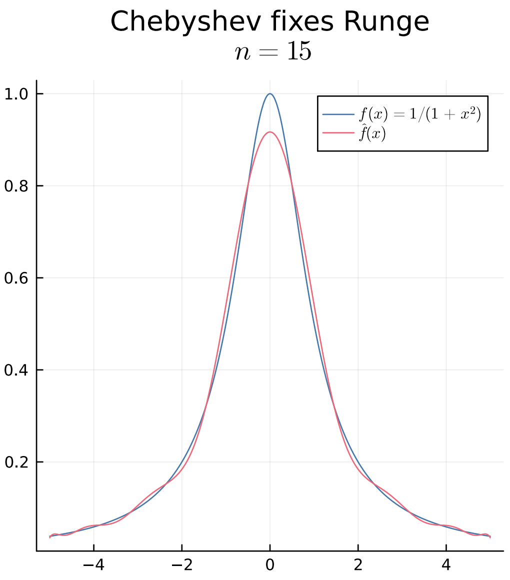

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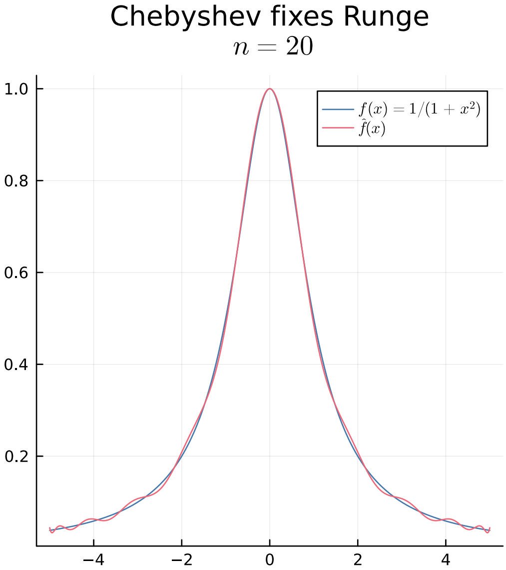

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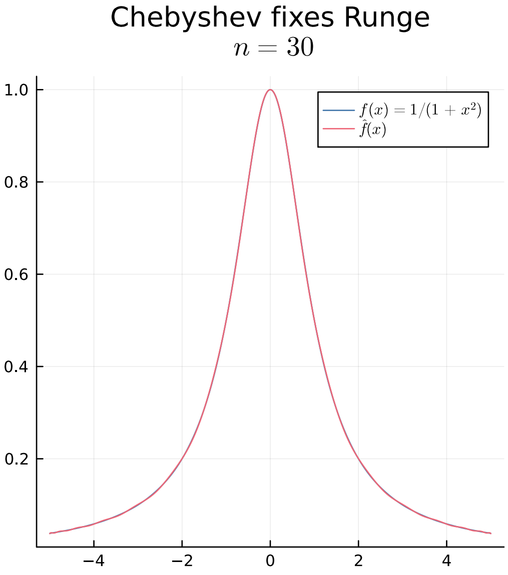

Chebyshev Interpolation

When to use?

- Chebyshev interpolation works really well if you are sure that your function is defined everywhere and smooth
- The smoother it is, the better Chebyshev approximation performs (faster convergence)
- Sometimes it has trouble at the boundary
  - This can be fixed by using a different set of points x_i that include the boundary node
  - This is called the **expanded Chebyshev array** – you can look this up if you need it
- The bigger trouble arises when you have functions that are not bounded: if you have a utility function that goes to $-\infty$ when $c \to 0$, this can cause serious problems for Chebyshev polynomials
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Section 3

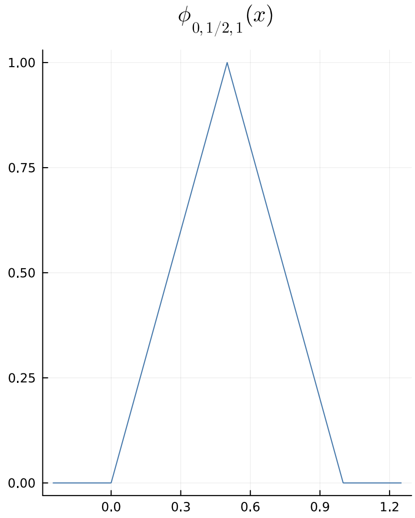
Linear Interpolation and Splines

Linear Interpolation

- Rather than use a family of polynomials that are defined everywhere, we can try polynomials that are more limited in scope
- In particular let's consider the piecewise linear functions (functions which look linear on any subinterval)
  - This is literally what you get if you just draw straight lines between the points on the graph
- Our prototypical piecewise linear function will be the "hat" function on $[x_1, x_2]$

$$\phi_{x_1,x_m,x_2}(x) = \begin{cases} \frac{x-x_1}{x_m-x_1} & \text{if } x_1 \le x \le x_m \\ 1 - \frac{x-x_m}{x_2-x_m} & \text{if } x_m < x \le x_2 \\ 0 & \text{otherwise} \end{cases}$$

- You can think of x_m as the point where ϕ attains its maximum value 1



Linear Interpolant

- Suppose we have a function $f : [a, b] \to \mathbb{R}$ and have the data points $\{(x_i, y_i)\}_{i=1}^{n}$.
- How do we construct our linear interpolant?
- Let's add up the appropriate "hat" functions:
  - For each i, let $\phi^i(x) = \phi_{x_{i-1},x_i,x_{i+1}}(x)$
  - This is putting a little hat function over every data point we have

    We have to be careful at the edges. Pick any $x_0 < x_1$ and $x_{n+1} > x_n$ so that this definition works
  - Take a look closely at ϕ^i. For all $1 < i < n$:

$$\phi^i(x_{i-1}) = 0 \qquad \phi^i(x_i) = 1 \qquad \phi^i(x_{i+1}) = 0$$

- Define $\hat{f}(x) := \sum_{i=1}^{n} c_i \phi^i(x)$ for some coefficients c_i
- Let's evaluate $\hat{f}$ at each x_j:

$$\hat{f}(x_j) = \underbrace{\sum_{i=1}^{n} c_i \phi^i(x_j)}_{\text{All 0 when } i \neq j} = c_j \phi^j(x_j) = c_j \tag{8}$$

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- How do we construct our linear interpolant?
- Let's add up the appropriate "hat" functions:
  - For each i, let $\phi^i(x) = \phi_{x_{i-1},x_i,x_{i+1}}(x)$
  - This is putting a little hat function over every data point we have

    We have to be careful at the edges. Pick any $x_0 < x_1$ and $x_{n+1} > x_n$ so that this definition works
  - Take a look closely at ϕ^i. For all $1 < i < n$:

$$\phi^i(x_{i-1}) = 0 \qquad \phi^i(x_i) = 1 \qquad \phi^i(x_{i+1}) = 0$$

- Define $\hat{f}(x) := \sum_{i=1}^n c_i\phi^i(x)$ for some coefficients c_i
- Let's evaluate $\hat{f}$ at each x_j:

$$\hat{f}(x_j) = \underbrace{\sum_{i=1}^{n} c_i\phi^i(x_j)}_{\text{All 0 when } i \neq j} = c_j\phi^j(x_j) = c_j \tag{8}$$

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Linear Interpolation Solves a Linear System

- Let's impose our interpolation conditions
  - We want $\hat{f}(x_i) = f(x_i) = y_i$ for all i
  - That means eq. (8) implies
$$y_i = \hat{f}(x_i) = c_i \quad \text{for all } i \tag{9}$$
- This is a (really simple) system of linear equations:
$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} \tag{10}$$
- It's almost trivial, but we're going to come back to this when we discuss splines

Linear Interpolation

When to use it?

- Linear interpolation is a great fallback if you have a badly behaved function
  - E.g., kinks, poles, etc...
- It's simple and easy to implement: it's basically our mental model anyway
- You'll never be confused about why it's doing what it's doing
- Downsides:
  - Slow convergence – you often need way more grid points to get a good approximation
  - Not differentiable at the data points – sometimes an optimizer will get stuck on a kink and you will get a poor solution

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Piecewise Cubic Approximation: Cubic Splines

- With piecewise linear functions, the problem is that they're not smooth enough
- What if we tried the same approach, but with a cubic polynomial on each sub-interval?
- Suppose for every interval $[x_{i-1}, x_i]$ we want our approximation to be a cubic polynomial:

$$\widehat{f}(x) = a_i + b_i x + c_i x^2 + d_i x^3 \qquad \text{for } x \in [x_{i-1}, x_i], \text{ and for all } i$$

- We have several conditions we want:

$$\text{Interpolation:} \qquad y_i = a_i + b_i x_i + c_i x_i^2 + d_i x_i^3 \quad \text{for } i = 1, \ldots, n \tag{11}$$

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Cubic splines solve a linear system

- So far this is all one big linear system of equations!
  - We know how to solve linear systems
  - Stack the conditions up in a matrix, and have the computer solve it
- We have $4n$ variables and $4n - 2$ equations
- Why did we lose two equations? Check back on the previous slide
  - Continuity of the derivatives is only imposed in the interior.
  - Need to make some assumptions about the derivatives of our approximation at the edges of our domain
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Spline Boundary Conditions

- There are three main options:
  - Natural spline: $\widehat{f}'(x_0) = 0 = \widehat{f}'(x_n)$
  - Hermite Spline: $\widehat{f}'(x_0) = y_0'$ and $\widehat{f}'(x_n) = y_n'$

    Assumes you know the true derivatives at the boundary
  - Secant spline: $\widehat{f}'(x_0) = \frac{\widehat{f}(x_1) - \widehat{f}(x_0)}{x_1 - x_0}$ and a similar condition for $\widehat{f}'(x_n)$

    Assumes a linear approximation of the derivative at the lower and upper bounds
- Which you choose depends on the specifics of the problem
  - Often the *natural* spline is not a good fit if you know your function is strictly concave (like a utility function)

How to actually use splines?

- Unless you're explicitly asked, don't code these up yourself
- Extremely efficient implementations (for splines and linear interpolation) are available in `Interpolations.jl`
- There are also some other fun things you can try:
  - Shape preserving splines: these splines add extra conditions to ensure that the approximation will never have a curvature that does not match the input data
  - I.e, if your data is sampled from a strictly concave function, the resulting spline will also be strictly concave
- Crucially, all of these methods generalize quite nicely to multiple dimensions:
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Section 4

Optional Content

Lagrange Interpolant is Unique

Theorem 1

Suppose we have data $\{(x_i, y_i) \mid i = 1, \ldots, n\}$ where the x_i are all unique. There is a unique polynomial of degree $n-1$ that interpolates these values.

Proof.

- Since the Vandermonde matrix V has full rank, we know that a solution $p(x)$ exists
- Suppose that $\hat{p}(x)$ is a polynomial of degree at most $n-1$ which also interpolates these points.
- We know that since $\hat{p}$ interpolates our data, $\hat{p}(x_i) = p(x_i) = y_i$ for all i.
- This means that $g(x) = p(x) - \hat{p}(x)$ is a polynomial of degree at most $n-1$ which has n distinct zeros (all of the data points).
- The only such polynomial is the zero polynomial, which implies that $p = \hat{p}$ $\square$

Back

Chebyshev Approximation Theorem

Theorem 2

Assume that $f : [-1, 1] \to \mathbb{R}$ *has continuous* k*th derivatives. If*

$$c_j \equiv \frac{2}{\pi} \int_{-1}^{1} \frac{f(x)T_j(x)}{\sqrt{1-x^2}} dx$$

and

$$C_n(x) \equiv \frac{1}{2} c_0 + \sum_{j=1}^{n} c_j T_j(x)$$

Then there is a $B < \infty$ *such that for all* $n \geq 2$*:*

$$\|f - C_n\|_\infty \leq \frac{B \log n}{n^k}$$

- This means that for smooth enough functions, our Chebyshev approximation will converge uniformly (and rapidly) to the true function
- Note that $\|f\|_\infty := \max_x |f(x)|$