Module 5 — The Marginalist Revolution

Utility, Optimisation, and the Birth of Neoclassical Economics

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Part I — The History

The Revolution of the 1870s

- In the early 1870s, three economists working **independently** arrived at the same idea almost simultaneously.
- They called it marginal utility, and it transformed economics from a study of classes and nations into a science of **individual choice**.
- One of the most remarkable cases of simultaneous discovery in the history of ideas.

The Three Founders

Jevons	Manchester, 1871
Menger	Vienna, 1871
Walras	Lausanne, 1874

William Stanley Jevons (1835–1882)

- English economist and logician, professor at Manchester.
- *The Theory of Political Economy* (1871): economics is "a calculus of pleasure and pain."
- Used mathematics and the language of **differential calculus** to formalise the idea of diminishing satisfaction.

"Value depends entirely upon utility . . . We have only to trace out carefully the natural laws of the variation of utility."

— Jevons (1871)

Key Idea

The value of a good is determined not by the total satisfaction it provides, but by the satisfaction from the last unit consumed — its **marginal utility**.

Menger and Walras

Carl Menger (1840–1921)

- Viennese lawyer turned economist.
- *Principles of Economics* (1871).
- Emphasised **subjective value**: goods have value because individuals value them, not because of embedded labour.
- Founded the **Austrian School**.

Léon Walras (1834–1910)

- Professor at Lausanne, Switzerland.
- *Elements of Pure Economics* (1874).
- Developed **general equilibrium theory**: all markets are interconnected and clear simultaneously.
- The most mathematically ambitious of the three.

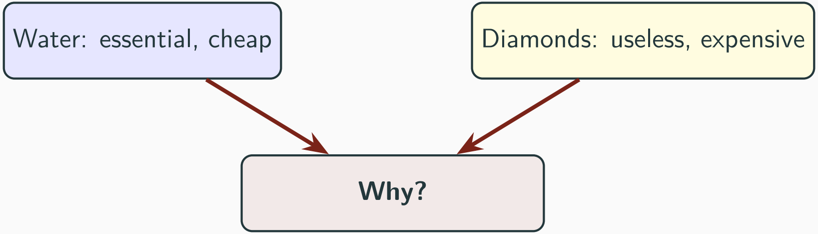
The Diamond-Water Paradox

The Classical Puzzle

Water is **essential for life** but cheap. Diamonds are **useless** but expensive.

If value comes from labour or usefulness, why is water cheap?

The classical economists — Smith, Ricardo, Marx — answered with the **labour theory of value**: goods are worth the labour required to produce them. But this never fully resolved the paradox, because water is *more* useful than diamonds.



The Marginalist Answer

Price reflects marginal utility, not *total* utility.

Water (abundant)

- Total utility: **very high**
- Marginal utility: **very low**
- The 1000th glass of water adds almost nothing.

Diamonds (scarce)

- Total utility: relatively low
- Marginal utility: **very high**
- One more diamond adds a lot.

It is not the first glass of water vs. the first diamond.
It is the *last* glass vs. the *last* diamond.

What Changed: Three Revolutions in One

The Marginalist Revolution reshaped economics in three fundamental ways:

1. **From classes to individuals.**
   Smith and Marx analysed workers, capitalists, and landlords as classes. The marginalists analysed individual consumers and firms making optimising decisions.
2. **From labour theory to subjective value.**
   Value became **subjective** — determined by individual preferences — rather than objective (embedded labour).
3. **Mathematics entered economics.**
   Jevons, Walras, and later Marshall expressed economic ideas as **functions, derivatives, and optimisation problems**.

Marshall and the Synthesis

- **Alfred Marshall** (1842–1924) synthesised the marginalists' insights into a coherent framework.
- *Principles of Economics* (1890): the standard textbook for half a century.
- Gave us the **supply-and-demand diagrams** still used today.
- His great achievement: deriving the demand curve from the theory of consumer choice.

Marshall's Method

Put mathematics in the footnotes. Present results with **diagrams and intuition**. Let the economics, not the algebra, be the star.

"Burn the mathematics."

Edinburgh Connection

The Scottish Tradition

Edinburgh's intellectual tradition — with its emphasis on systematic, evidence-based inquiry going back to the Scottish Enlightenment — shaped the reception of marginalism in Britain.

- Marshall acknowledged debts to the **Scottish philosophical tradition** and its empirical method.
- Edinburgh's economics department adopted the new mathematical methods **earlier than many British universities**.
- The marginalists' insistence on **observation and formal reasoning** echoed the values of Hume, Smith, and the Enlightenment.

From Smith's verbal reasoning about the invisible hand to Jevons's "calculus of pleasure and pain" — the Scottish tradition of rigour endures.

Part II — The Computation

Setting Up

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.optimize import minimize
```

- `numpy` — numerical arrays and operations
- `matplotlib` — plotting
- `minimize` — constrained optimisation
  (the computational equivalent of choosing the best affordable bundle)

Utility and Diminishing Marginal Utility

A consumer gets satisfaction — **utility** — from consuming goods. The key insight of the marginalists:

Utility increases with consumption but at a decreasing rate.

A common utility function:

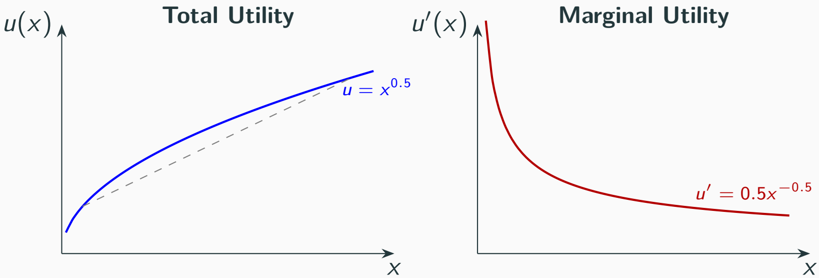
$$u(x) = x^{0.5}$$

The **marginal utility** is the derivative:

$$u'(x) = 0.5 \cdot x^{-0.5}$$

- $u'(x) > 0$: more is always better
- $u''(x) < 0$: but each additional unit adds less than the one before
- This is Jevons's **law of diminishing marginal utility**

Total vs. Marginal Utility



The first slice of cake is heavenly; the tenth is merely okay; the twentieth might make you ill.

Plotting Utility in Python

```
x = np.linspace(0.01, 10, 200)

fig, axes = plt.subplots(1, 2, figsize=(12, 4.5))

# Total utility
axes[0].plot(x, x**0.5, color='steelblue', linewidth=2.5)
axes[0].set_xlabel('Quantity consumed')
axes[0].set_ylabel('Utility u(x)')
axes[0].set_title('Total Utility: Increasing but Concave')

# Marginal utility
axes[1].plot(x, 0.5 * x**(-0.5), color='coral', linewidth=2.5)
axes[1].set_xlabel('Quantity consumed')
axes[1].set_ylabel("Marginal utility u'(x)")
axes[1].set_title('Marginal Utility: Positive but Falling')

plt.tight_layout()
plt.show()
```

The Diamond-Water Paradox: A Numerical Example

```
# u(x) = x^0.5
water_quantity = 100     # abundant
diamond_quantity = 0.5   # scarce

total_u_water = water_quantity**0.5          # 10.00
marginal_u_water = 0.5 * water_quantity**(-0.5)    # 0.05

total_u_diamond = diamond_quantity**0.5      #  0.71
marginal_u_diamond = 0.5 * diamond_quantity**(-0.5) # 0.71
```

	Water ($x = 100$)	Diamonds ($x = 0.5$)
Total utility	10.00 **(high)**	0.71
Marginal utility	0.05	0.71 **(high)**

Price reflects marginal utility, not total utility.

That is why diamonds cost more than water.

Consumer Choice: Two Goods and a Budget

The marginalists' core problem:

$$\max_{x_1, x_2} \; u(x_1, x_2) \quad \text{subject to} \quad p_1 x_1 + p_2 x_2 \leq m$$

We use the **Cobb-Douglas utility function**:

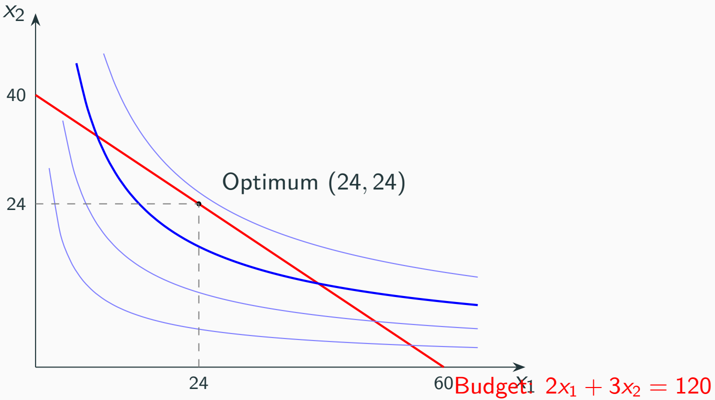
$$u(x_1, x_2) = x_1^a \cdot x_2^{1-a}$$

- a governs how much the consumer values good 1 vs. good 2.
- Parameters: $a = 0.4$, $p_1 = 2$, $p_2 = 3$, $m = 120$.

Analytical Solution (Cobb-Douglas)

$$x_1^* = \frac{a \cdot m}{p_1} = \frac{0.4 \times 120}{2} = 24, \qquad x_2^* = \frac{(1-a) \cdot m}{p_2} = \frac{0.6 \times 120}{3} = 24$$

Indifference Curves and the Budget Line



The consumer reaches the **highest indifference curve** that still touches the budget constraint — the tangency point.

The Tangency Condition

At the optimum, the slope of the indifference curve equals the slope of the budget line:

$$\underbrace{\frac{MU_1}{MU_2}}_{\text{Marginal rate of substitution}} = \underbrace{\frac{p_1}{p_2}}_{\text{Price ratio}}$$

- **MRS** $= \frac{a}{1-a} \cdot \frac{x_2}{x_1}$ for Cobb-Douglas.
- At the optimum: $\frac{0.4}{0.6} \cdot \frac{24}{24} = \frac{2}{3} = \frac{p_1}{p_2}$ ✓
- Intuition: the rate at which the consumer is *willing* to trade goods equals the rate at which the market *allows* it.

Solving with `scipy.optimize.minimize`

Since `minimize` minimises, we minimise **negative** utility:

```python
a, p1, p2, m = 0.4, 2, 3, 120

def neg_utility(x):
    x1, x2 = x
    if x1 <= 0 or x2 <= 0:
        return 1e10  # penalty for non-positive consumption
    return -(x1**a * x2**(1 - a))

budget = {'type': 'ineq',
          'fun': lambda x: m - p1*x[0] - p2*x[1]}
bounds = [(0.01, None), (0.01, None)]

result = minimize(neg_utility, x0=[10, 10],
                  method='SLSQP',
                  bounds=bounds, constraints=budget)

x1_star, x2_star = result.x
print(f"x1* = {x1_star:.2f}, x2* = {x2_star:.2f}")
# x1* = 24.00, x2* = 24.00
```

Verifying the Solution

```python
u_star = x1_star**a * x2_star**(1 - a)

print(f"Optimal bundle:")
print(f"  x1* = {x1_star:.2f} (spending {p1*x1_star:.0f})")
print(f"  x2* = {x2_star:.2f} (spending {p2*x2_star:.0f})")
print(f"  Total spending: {p1*x1_star + p2*x2_star:.0f}")
print(f"  Utility: {u_star:.4f}")

# Analytical solution for Cobb-Douglas
x1_a = a * m / p1      # = 24
x2_a = (1-a) * m / p2  # = 24
print(f"\nAnalytical: x1* = {x1_a}, x2* = {x2_a}")
```

Cobb-Douglas Property

The consumer spends exactly fraction a of income on good 1 and $(1-a)$ on good 2, **regardless of prices**.

$$p_1 x_1^* = a \cdot m = 0.4 \times 120 = 48, \qquad p_2 x_2^* = (1-a) \cdot m = 0.6 \times 120 = 72$$

Deriving the Demand Curve

Marshall's great achievement: deriving the demand curve from consumer choice.

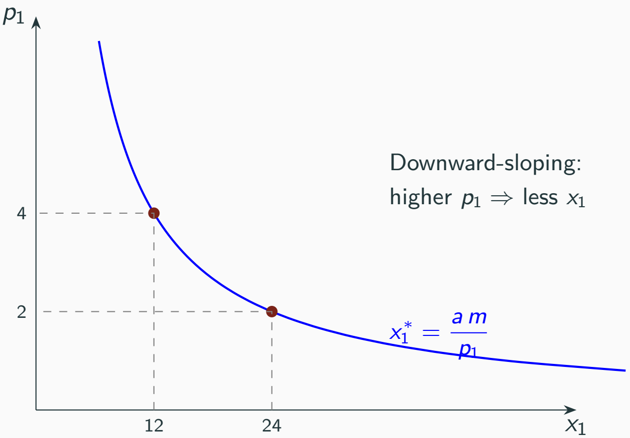
As p_1 rises, the consumer buys less of good 1.

```
p1_range = np.linspace(0.5, 10, 30)
x1_demand = []

for p1_val in p1_range:
    cons = {'type': 'ineq',
            'fun': lambda x, p=p1_val: m - p*x[0] - p2*x[1]}
    res = minimize(neg_utility, x0=[10, 10], method='SLSQP',
                   bounds=[(0.01, None), (0.01, None)],
                   constraints=cons)
    x1_demand.append(res.x[0])

# Analytical demand: x1* = a*m/p1
x1_analytical = a * m / p1_range
```

The demand curve slopes downward — exactly as Marshall drew it in 1890.

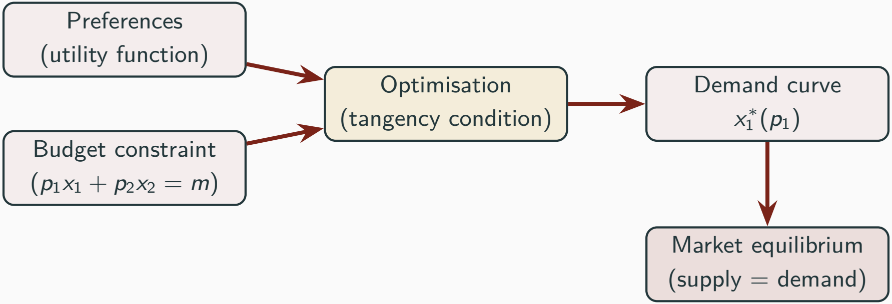
But now we have **derived** it from individual optimisation, not assumed it.

The Demand Curve



A higher p_1 has both an **income effect** (consumer is poorer) and a **substitution effect** (good 1 is relatively more expensive), both reducing demand for a normal good.

From Utility to Markets: The Big Picture



The marginalists provided microfoundations for Smith's supply-and-demand model: individual rational choice generates the demand curve, which then determines market equilibrium.

Part III — Exercises & Discussion

Exercise 1: Income and Substitution Effects

When the price of good 1 rises, two things happen:

1. The consumer is effectively **poorer** (income effect).
2. Good 1 is now **relatively more expensive** (substitution effect).

Using $a = 0.4$, $p_2 = 3$, $m = 120$:

(a) Compute x_1^* at $p_1 = 2$ and at $p_1 = 4$. What is Δx_1?

(b) **Compensated demand:** at $p_1 = 4$, how much income would keep utility at the old level? Use `minimize` to find the minimum expenditure that achieves u^*_{old}.

(c) Compute the substitution effect and income effect separately. Verify they sum to Δx_1.

(d) For a **normal good**, both effects reduce x_1. Verify.

Exercise 2: Beyond Cobb-Douglas — CES Utility

The **CES** utility function allows different degrees of substitutability:

$$u(x_1, x_2) = \left(a \cdot x_1^{\rho} + (1 - a) \cdot x_2^{\rho}\right)^{1/\rho}$$

where $\rho = (\sigma - 1)/\sigma$ and σ is the elasticity of substitution.

(a) Define a CES utility function. Using $a = 0.5$, $p_1 = 2$, $p_2 = 3$, $m = 120$, solve for $\sigma = 0.5, 1, 2, 5$.

(b) Derive the demand curve for each σ (vary p_1 from 0.5 to 8). Plot all four on one graph. Which σ gives the most elastic demand?

(c) What happens as $\sigma \to \infty$? (Hint: perfect substitutes.) What does the demand curve look like?

Exercise 3: From Smith to the Marginalists

In Module 1 we found market equilibrium by setting supply equal to demand. Now we can *derive* the demand curve from first principles.

(a) 100 identical consumers each have $m = 120$ and Cobb-Douglas utility with $a = 0.4$. Compute the **aggregate demand curve**: total demand $= 100 \times x_1^*(p_1)$.

(b) Define a linear supply: $Q^S(p_1) = 500 + 200\,p_1$.

(c) Use `fsolve` to find the market equilibrium. Plot supply and demand.

(d) In 2–3 sentences, explain how the marginalist approach provides microfoundations for the supply-and-demand model Smith described verbally.

Quiz

Quiz: Conceptual Questions

1. The diamond-water paradox is resolved by recognising that price reflects:
   (a) Total utility (b) Marginal utility (c) Labour content (d) Production cost
2. The Marginalist Revolution shifted economics from:
   (a) Trade to monetary theory (b) Class-based analysis to individual optimisation
   (c) Mathematics to verbal reasoning (d) Micro to macroeconomics
3. Diminishing marginal utility means:
   (a) Total utility decreases
   (b) Each additional unit provides less additional satisfaction
   (c) Utility is negative beyond some point (d) Consumers prefer less to more
4. Marshall's main contribution was:
   (a) Inventing calculus
   (b) Synthesising marginalist theory into supply-and-demand
   (c) Disproving comparative advantage (d) Founding the LSE

Quiz: Computational Questions

5. The consumer's optimal bundle is where:
   (a) All income goes to the cheaper good
   (b) The highest indifference curve touches the budget constraint
   (c) Marginal utility is zero (d) Budget constraint crosses the origin
6. For $u = x_1^{0.4} x_2^{0.6}$ with $p_1 = 2$, $p_2 = 3$, $m = 120$, the optimal x_1^* is:
   (a) $0.4 \times 120/2 = 24$ (b) 36 (c) 60 (d) 16
7. To maximise utility using `minimize`, we minimise:
   (a) Utility directly (b) Negative utility (c) The budget constraint (d) Price times quantity
8. For Cobb-Douglas utility, the share of income spent on good 1 is:
   (a) Always 50% (b) Equal to the exponent a (c) Depends on the price ratio (d) 1 minus the price ratio

Key Takeaways

What We Learned This Week

1. The **Marginalist Revolution** (1870s) shifted economics from class-based, labour-theory reasoning to **individual optimisation** with subjective value.
2. The **diamond-water paradox** dissolves once we distinguish *total* from *marginal* utility.
3. Consumers maximise utility subject to a **budget constraint**; the solution is the tangency between the highest indifference curve and the budget line.
4. `scipy.optimize.minimize` with SLSQP solves constrained optimisation problems — minimise negative utility.
5. The **demand curve** can be *derived* from utility maximisation, not just assumed — providing microfoundations for Smith's supply-and-demand intuition.

Further Reading

- Jevons, W.S. *The Theory of Political Economy* (1871), Ch. 3.
- Backhouse, R. *The Penguin History of Economics*, Ch. 7 ("The Marginalist Revolution").
- Marshall, A. *Principles of Economics* (1890), Book III (on demand and utility).
- Mas-Colell, A., Whinston, M., & Green, J. *Microeconomic Theory*, Ch. 3 (for the mathematically adventurous).

Next week: Keynes and the Great Depression —
why the marginalists' beautiful model of rational individuals
could not explain mass unemployment.