Module 7 — The Solow Growth Model

Capital Accumulation, Steady States, and the Sources of Growth

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From Smith to Simulation

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Part I — The History

Post-War Prosperity and an Unanswered Question

After 1945 the world economy experienced unprecedented growth:

- Western Europe rebuilt from rubble in a single generation.
- The US doubled living standards between 1945 and 1970.
- Japan rose from wartime devastation to the world's second-largest economy.
- West Germany's *Wirtschaftswunder* tripled real GDP per capita (1950–1970).

The Open Question

Where did all this growth come from? Smith, Ricardo, and Marx had theories of value but **no formal model of how economies grow over time**. Keynes explained fluctuations but said little about the long-run trajectory.

The Harrod–Domar Problem

In the late 1940s, Roy Harrod and Evsey Domar independently proposed growth models:

- Growth depends on the **saving rate** and the **capital–output ratio**.
- But the models had a fatal flaw: the growth path was knife-edge unstable.
- Any deviation from the "warranted rate" led to ever-widening booms or slumps.
- No diminishing returns $\Rightarrow$ no self-correcting mechanism.

The economics profession needed a model where growth was **stable** — where the economy would naturally settle at a sustainable path.

Robert Solow and the 1956 Paper

In 1956, **Robert Solow** (1924–2023) at MIT published *"A Contribution to the Theory of Economic Growth"*.

1. Output is produced from **capital** and **labour** via a production function.
2. A fixed fraction of output is **saved and invested**.
3. Capital **depreciates** — machines wear out.
4. The production function exhibits diminishing returns to capital.

The Startling Conclusion

Capital accumulation **alone** cannot sustain long-run growth. Because of diminishing returns, each new unit of capital adds less output. The economy settles at a steady state.

Solow's Key Insight

"All theory depends on assumptions which are not quite true. That is what makes it theory. The art of successful theorizing is to make the inevitable simplifying assumptions in such a way that the final results are not very sensitive."

— Robert Solow, *Quarterly Journal of Economics* (1956)

- By adding **diminishing returns**, Solow turned Harrod–Domar's knife-edge into a globally stable steady state.
- The model is simple enough to solve analytically, yet rich enough to explain the major facts of post-war growth.

The Solow Residual: Technology as the True Engine

Solow's empirical companion paper (1957) decomposed US growth (1909–1949):

The Decomposition

- Capital and labour together explained only $\approx$**12.5%** of output growth.
- The remaining 87.5% was a **residual**.

Total Factor Productivity

The residual captures:

- Technological progress
- Better management
- Improved institutions
- Education and innovation

Solow's message: *technology, not saving, is the true engine of long-run growth.*

The Growth Miracles

Solow's framework explains the post-war miracles:

Germany & Japan Started with capital stocks destroyed but human capital intact. Low $k \Rightarrow$ enormous returns to investment $\Rightarrow$ rapid convergence toward steady state.

Asian Tigers South Korea, Taiwan, Hong Kong, Singapore followed a similar path from the 1960s onward.

The model also predicted that rapid growth would slow as countries approached their steady states — which is precisely what happened.

Edinburgh Connection and the Nobel Prize

- The Scottish Enlightenment's commitment to **empirical inquiry** — from Hume's insistence on observation to Smith's systematic evidence — laid the groundwork for quantitative growth economics.
- Solow's method of **confronting theory with data**, decomposing growth into measurable components, and letting the residual speak for itself embodies this empiricist tradition.
- Today, Edinburgh's economics department continues this tradition through research on productivity and innovation.

Nobel Prize 1987

Robert Solow received the Nobel Memorial Prize in Economic Sciences in 1987 "for his contributions to the theory of economic growth."

Part II — The Computation

Setting Up

```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.optimize import fsolve
```

- `numpy` — numerical arrays and operations
- `matplotlib` — plotting
- `fsolve` — numerical root-finder (finds the steady state where $dk/dt = 0$)

The Cobb–Douglas Production Function

Solow used a **Cobb–Douglas** production function:

$$Y = A \cdot K^{\alpha} \cdot L^{1-\alpha}$$

- Y = total output (GDP), K = capital stock, L = labour force
- A = total factor productivity (technology)
- α = capital share of output (typically $\approx 1/3$)

Dividing by L, with $y = Y/L$ and $k = K/L$:

$$\boxed{y = A \cdot k^{\alpha}}$$

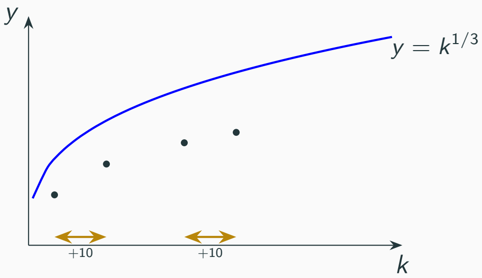
This is the intensive form. Since $\alpha < 1$, doubling capital per worker **less than doubles** output per worker — **diminishing returns**.

Diminishing Returns in Python

```python
A = 1.0          # total factor productivity
alpha = 1/3      # capital share

def production(k, A=A, alpha=alpha):
    """Output per worker: y = A * k^alpha."""
    return A * k**alpha

# From k=5 to k=15 (adding 10): output rises by 0.740
# From k=30 to k=40 (adding 10): output rises by 0.258
# Same extra capital, LESS extra output.
```



The Fundamental Solow Equation

Capital per worker evolves according to:

$$\boxed{\frac{dk}{dt} = s \cdot A \cdot k^{\alpha} - \delta \cdot k}$$

- s = saving rate (fraction of output saved and invested)
- δ = depreciation rate (fraction of capital that wears out)

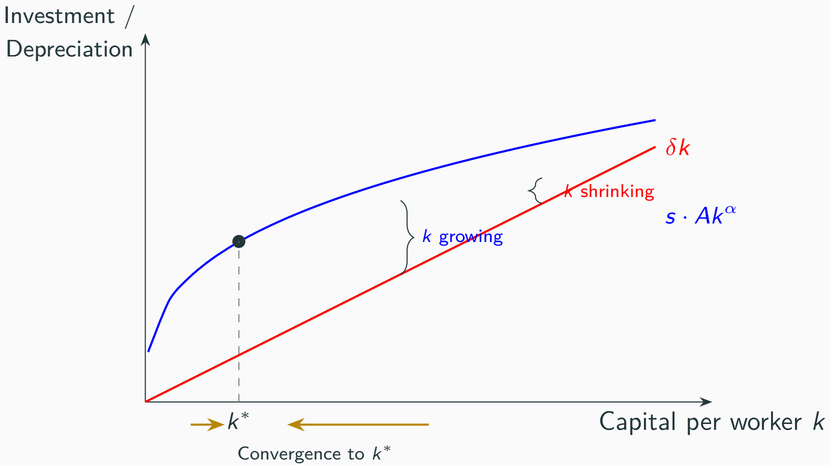
Two opposing forces:

Investment $s \cdot A \cdot k^{\alpha}$ — adds to capital stock (concave, diminishing returns)

Depreciation $\delta \cdot k$ — erodes capital stock (linear in k)

When investment = depreciation: k is constant $\Rightarrow$ the steady state.

The Solow Diagram



Below k^*: investment $>$ depreciation $\Rightarrow$ capital grows.
Above k^*: depreciation $>$ investment $\Rightarrow$ capital shrinks.

The Solow Diagram in Python

```
s = 0.3          # saving rate
delta = 0.05     # depreciation rate

k_vals = np.linspace(0, 80, 300)
investment = s * production(k_vals)
depreciation = delta * k_vals

# Steady state: where the curves cross
k_star = (s * A / delta) ** (1 / (1 - alpha))
y_star = production(k_star)
print(f"Steady state: k* = {k_star:.2f}, y* = {y_star:.2f}")
# Output: k* = 14.70, y* = 2.45
```

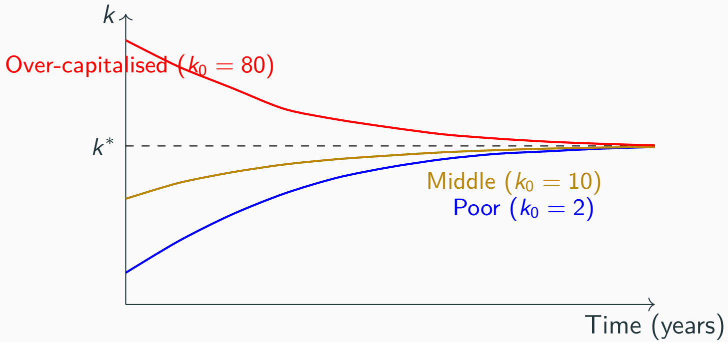
At steady state $k^* \approx 14.70$:

- Investment $= s \cdot y^* = 0.3 \times 2.45 = 0.735$
- Depreciation $= \delta \cdot k^* = 0.05 \times 14.70 = 0.735$
- They are exactly equal — capital per worker is constant.

Simulating Convergence

We simulate with Euler integration:

$$k_{t+1} = k_t + \Delta t \cdot \left(s \cdot A \cdot k_t^{\alpha} - \delta \cdot k_t\right)$$



Regardless of starting point, all paths converge to k^*.

Convergence Simulation in Python

```
def simulate_solow(k0, s, delta, A, alpha, T=200, dt=1.0):
    """Simulate the Solow model from initial capital k0."""
    k = np.zeros(T)
    k[0] = k0
    for t in range(T - 1):
        y = A * k[t]**alpha
        dk = s * y - delta * k[t]
        k[t + 1] = k[t] + dt * dk
    y = A * k**alpha
    c = (1 - s) * y
    return k, y, c

# Simulate from different starting points
for k0 in [2.0, 10.0, 80.0]:
    k, y, c = simulate_solow(k0, s, delta, A, alpha, T=200)
    print(f"k0={k0:5.1f} -> k_final={k[-1]:.2f}")
```

All three paths converge to $k^* \approx 14.70$:

the fundamental stability property of the Solow model.

Finding the Steady State Analytically

At steady state, $dk/dt = 0$:

$$s \cdot A \cdot k^{*\alpha} = \delta \cdot k^*$$

Solving:

$$k^{*\,1-\alpha} = \frac{s \cdot A}{\delta} \qquad \Longrightarrow \qquad \boxed{k^* = \left(\frac{s \cdot A}{\delta}\right)^{\frac{1}{1-\alpha}}}$$

With $s = 0.3$, $A = 1$, $\alpha = 1/3$, $\delta = 0.05$:

$$k^* = \left(\frac{0.3}{0.05}\right)^{3/2} = 6^{3/2} = 6\sqrt{6} \approx 14.70$$

$$y^* = k^{*1/3} \approx 2.45, \qquad c^* = (1 - s)\, y^* \approx 1.71$$

Finding the Steady State with `fsolve`

```python
# Analytical
k_star_analytical = (s * A / delta) ** (1 / (1 - alpha))
print(f"Analytical: k* = {k_star_analytical:.6f}")

# Numerical with fsolve
def solow_equation(k):
    """dk/dt = s*A*k^alpha - delta*k. Zero at steady state."""
    return s * A * k**alpha - delta * k

k_star_numerical = fsolve(solow_equation, x0=10.0)[0]
print(f"Numerical:  k* = {k_star_numerical:.6f}")
print(f"Match: {np.allclose(k_star_analytical, k_star_numerical)}")
```

- `fsolve` finds where $f(k) = 0$.
- Numerical and analytical solutions agree to **machine precision**.
- The numerical approach generalises to models where no closed-form solution exists.

Comparative Statics: How Parameters Shape the Steady State

$$k^* = \left(\frac{s \cdot A}{\delta}\right)^{\frac{1}{1-\alpha}}$$

Saving rate s $\uparrow$

Higher $s \Rightarrow$ higher k^* and y^*.

But consumption c^* first rises, then falls (saving "too much").

Depreciation δ $\uparrow$

Higher $\delta \Rightarrow$ lower k^*, y^*, c^*.

Capital wears out faster; the economy is poorer.

Technology A $\uparrow$

Higher $A \Rightarrow$ higher k^*, y^*, c^*.

The only source of sustained improvement.

Comparative Statics in Python

```
# Vary saving rate
s_range = np.linspace(0.05, 0.60, 100)
k_star_s = (s_range * A / delta) ** (1 / (1 - alpha))
y_star_s = A * k_star_s**alpha
c_star_s = (1 - s_range) * y_star_s

# Vary depreciation rate
delta_range = np.linspace(0.01, 0.15, 100)
k_star_d = (s * A / delta_range) ** (1 / (1 - alpha))

# Vary technology
A_range = np.linspace(0.5, 3.0, 100)
k_star_A = (s * A_range / delta) ** (1 / (1 - alpha))
```

Key insight: a higher saving rate raises the *level* of output but **not** the long-run *growth rate* (which is zero in the basic model without technological progress).

The Golden Rule Saving Rate

Steady-state consumption:

$$c^* = y^* - \delta\, k^* = A\, k^{*\alpha} - \delta\, k^*$$

Maximise with respect to k^*:

$$\frac{dc^*}{dk^*} = \alpha A\, k^{*\,\alpha-1} - \delta = 0$$

Golden Rule capital:

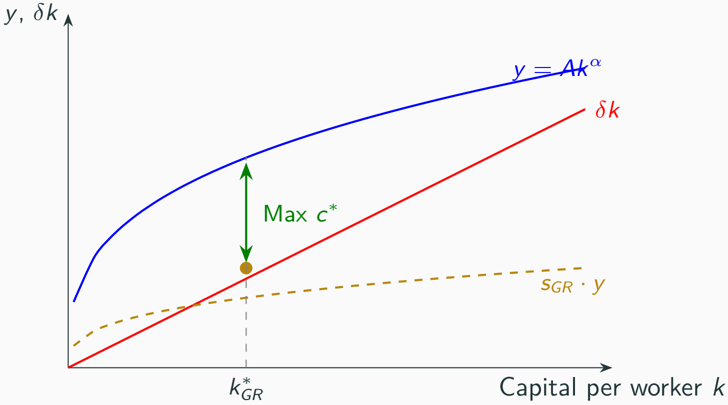
$$k^*_{GR} = \left(\frac{\alpha A}{\delta}\right)^{\frac{1}{1-\alpha}}$$

Golden Rule saving rate:

$$\boxed{s_{GR} = \alpha}$$

The saving rate that maximises consumption equals the **capital share of output**.

The Golden Rule Diagram



At the Golden Rule, the gap between y and δk is **largest** — consumption per worker is maximised.

Golden Rule in Python

```python
s_golden = alpha  # Golden Rule saving rate = capital share
k_golden = (s_golden * A / delta) ** (1 / (1 - alpha))
y_golden = A * k_golden**alpha
c_golden = (1 - s_golden) * y_golden

print(f"Golden Rule: s_GR = {s_golden:.4f}")
print(f"  k*_GR = {k_golden:.4f}")
print(f"  y*_GR = {y_golden:.4f}")
print(f"  c*_GR = {c_golden:.4f}")
```

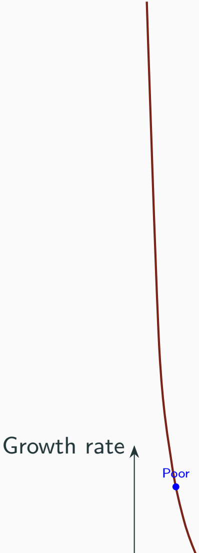
Over-saving

With $s = 0.30 > s_{GR} = 0.333$? Actually $s < s_{GR}$ here, so we under-save slightly. But if a country saves *more* than α, it is **dynamically inefficient**: it could consume more in *every* period by saving less.

Convergence: Poor Countries Grow Faster

One of the Solow model's most powerful predictions: conditional convergence.

- Countries far from k^* have **high** marginal returns to capital $\Rightarrow$ grow fast.
- Countries near k^* have **low** marginal returns $\Rightarrow$ grow slowly.
- This explains the post-war miracles of Germany, Japan, and the Asian Tigers.



Growth Accounting

Starting from $Y = AK^{\alpha}L^{1-\alpha}$, taking logs and differentiating:

$$\frac{\Delta Y}{Y} = \underbrace{\frac{\Delta A}{A}}_{\text{TFP growth}} + \alpha \cdot \frac{\Delta K}{K} + (1-\alpha) \cdot \frac{\Delta L}{L}$$

The **Solow residual** (TFP growth) is computed as:

$$\frac{\Delta A}{A} = \frac{\Delta Y}{Y} - \alpha \cdot \frac{\Delta K}{K} - (1-\alpha) \cdot \frac{\Delta L}{L}$$

- Everything we *can* measure (capital, labour) goes on the right.
- Everything we *cannot* measure (technology, institutions, management) is captured by the residual.
- Solow (1957): the residual accounted for 87.5% of US growth.

The Population Growth Extension

With labour force growth at rate n, capital is diluted:

$$\frac{dk}{dt} = s \cdot A \cdot k^{\alpha} - (\delta + n) \cdot k$$

The **break-even investment** line becomes steeper:

$$k^* = \left(\frac{s \cdot A}{\delta + n} \right)^{\frac{1}{1-\alpha}}$$

- Higher population growth $n \Rightarrow$ lower k^* and y^*.
- Capital must be spread among more workers — "capital dilution."
- Countries with rapid population growth tend to have **lower** living standards (per capita), consistent with data.
- The Golden Rule saving rate remains $s_{GR} = \alpha$.

Part III — Exercises

Exercise 1: Two-Country Convergence

Two countries share $A = 1$, $\alpha = 1/3$, $s = 0.25$, $\delta = 0.05$ but start at different capital levels.

1. Compute the common steady state k^*, y^*, c^*.
2. Simulate both countries for 150 years from $k_0^{\text{poor}} = 2$ (war-ravaged Germany) and $k_0^{\text{rich}} = 20$ (the United States). Plot k and y over time.
3. Compute and plot the annual growth rate of y for both. When does the poor country's growth rate first fall below 1%?
4. Define the *catch-up ratio* $y_{\text{poor}}/y_{\text{rich}}$. In approximately what year does the poor country reach 90% of the rich country's output per worker?

Exercise 2: Growth Accounting with Data

Decade	$\Delta Y/Y$	$\Delta K/K$	$\Delta L/L$
1950s	7.5%	9.0%	2.0%
1960s	6.0%	8.0%	1.5%
1970s	4.0%	5.5%	1.0%
1980s	3.0%	4.0%	1.0%
1990s	2.5%	3.0%	0.5%

Use $\alpha = 1/3$. For each decade:

1. Compute the contributions of capital, labour, and TFP growth.
2. Create a stacked bar chart of the three contributions.
3. In which decade was TFP growth highest?
4. This pattern is characteristic of which real-world countries?

Exercise 3: Population Growth Extension

With population growth at rate n:

$$\frac{dk}{dt} = s \cdot A \cdot k^{\alpha} - (\delta + n) \cdot k$$

Use $A = 1$, $\alpha = 1/3$, $s = 0.3$, $\delta = 0.05$.

1. Derive and compute k^*, y^*, c^* for $n = 0.02$. Verify with `fsolve`.
2. Plot the Solow diagram with $(\delta + n)k$ and δk lines on the same graph.
3. Simulate for $n = 0$, $n = 0.02$, $n = 0.04$. What does higher population growth do to steady-state living standards?
4. Derive the Golden Rule when $n > 0$. Why does n not affect s_{GR}?

Key Takeaways

Quiz: Test Your Understanding

1. Why does capital accumulation alone fail to sustain long-run growth?
   Diminishing returns: each additional unit of capital adds less output.
2. What does the Solow residual measure?
   The fraction of output growth **not explained** by capital and labour growth — i.e., TFP.
3. At the steady state, what condition holds?
   Investment **exactly replaces** depreciated capital: $s \cdot y^* = \delta \cdot k^*$.
4. What is the Golden Rule saving rate?
   $s_{GR} = \alpha$ — the capital share of output.
5. How does the Solow model explain post-war growth miracles?
   Countries far below k^* enjoy high returns to capital $\Rightarrow$ rapid convergence.

What We Learned This Week

1. Robert Solow (1956) showed that **capital accumulation** with diminishing returns leads to a stable steady state.
2. The **Solow residual** (TFP) accounts for most of observed growth — technology, not saving, is the true engine.
3. The model predicts **conditional convergence**: poor countries grow faster when they share the same fundamentals.
4. The **Golden Rule** $s_{GR} = \alpha$ maximises steady-state consumption.
5. We can **simulate** the Solow model, find steady states with `fsolve`, and perform **growth accounting** to decompose the sources of growth.

Further Reading

- Solow, R.M. "A Contribution to the Theory of Economic Growth," *Quarterly Journal of Economics* (1956).
- Solow, R.M. "Technical Change and the Aggregate Production Function," *Review of Economics and Statistics* (1957).
- Jones, C.I. *Introduction to Economic Growth*, Chapters 2–3.
- Romer, D. *Advanced Macroeconomics*, Chapter 1.
- Warsh, D. *Knowledge and the Wealth of Nations* (narrative history of growth theory).

Next week: Beyond the Solow Model —
If technology drives growth, where does technology come from?
Endogenous growth theory and the economics of innovation.